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2007 J. Phys. A: Math. Theor. 40 7473

(http://iopscience.iop.org/1751-8121/40/27/005)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 7473–7508 doi:10.1088/1751-8113/40/27/005

Regular XXZ Bethe states at roots of unity as highest weight vectors of the sl_2 loop algebra

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Received 17 March 2007, in final form 11 May 2007
Published 20 June 2007
Online at stacks.iop.org/JPhysA/40/7473

Abstract

We show that every regular Bethe ansatz eigenvector of the XXZ spin chain at roots of unity is a highest weight vector of the sl_2 loop algebra, for some restricted sectors with respect to eigenvalues of the total spin operator S^Z, and evaluate explicitly the highest weight in terms of the Bethe roots. We also discuss whether a given regular Bethe state in the sectors generates an irreducible representation or not. In fact, we present such a regular Bethe state in the inhomogeneous case that generates a reducible Weyl module. Here, we call a solution of the Bethe ansatz equations which is given by a set of distinct and finite rapidities *regular Bethe roots*. We call a nonzero Bethe ansatz eigenvector with regular Bethe roots a *regular Bethe state*.

PACS numbers: 75.10.Pq, 75.10.Jm, 05.50.+q

1. Introduction

The XXZ spin chain is one of the most important exactly solvable quantum systems. The Hamiltonian under the periodic boundary conditions is given by

$$\mathcal{H}_{\mathrm{XXZ}} = \frac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^X\sigma_{j+1}^X + \sigma_j^Y\sigma_{j+1}^Y + \Delta\sigma_j^Z\sigma_{j+1}^Z\right). \tag{1.1}$$

Here we define parameter q from the XXZ coupling Δ by $\Delta = (q+q^{-1})/2$. When q is a root of unity, the XXZ Hamiltonian commutes with the generators of the sl_2 loop algebra [14]. Let q_0 be a root of unity satisfying $q_0^{2N} = 1$ for an integer N. We introduce operators $S^{\pm(N)}$ as follows:

$$S^{\pm(N)} = \sum_{1\leqslant j_1<\cdots<j_N\leqslant L} q_0^{\frac{N}{2}\sigma^Z}\otimes\cdots\otimes q_0^{\frac{N}{2}\sigma^Z}\otimes\sigma_{j_1}^{\pm}\otimes q_0^{\frac{(N-2)}{2}\sigma^Z}\otimes\cdots\otimes q_0^{\frac{(N-2)}{2}\sigma^Z}$$
$$\otimes\sigma_{j_2}^{\pm}\otimes q_0^{\frac{(N-4)}{2}\sigma^Z}\otimes\cdots\otimes\sigma_{j_N}^{\pm}\otimes q_0^{-\frac{N}{2}\sigma^Z}\otimes\cdots\otimes q_0^{-\frac{N}{2}\sigma^Z}. \tag{1.2}$$

They are derived from the Nth power of the generators $S^{\pm}$ of the quantum group $U_q(sl_2)$. We define $T^{(\pm)}$ by the complex conjugates of $S^{\pm(N)}$, i.e. $T^{\pm(N)} = (S^{\pm(N)})^*$. The operators, $S^{\pm(N)}$ and $T^{\pm(N)}$, generate the sl_2 loop algebra, $U(L(sl_2))$. Let us consider sectors with respect to eigenvalues of the total spin operator S^Z. We call the sector $S^Z \equiv 0 \pmod N$ *sector A*. Here the value of S^Z is given by an integral multiple of N. It was shown that the following commutation relations hold in sector A [14]:

$$[S^{\pm(N)}, \mathcal{H}_{XXZ}] = [T^{\pm(N)}, \mathcal{H}_{XXZ}] = 0. \tag{1.3}$$

Let us assume that rapidities, $t_1, t_2, \ldots, t_R$, satisfy the Bethe ansatz equations at a given value of q as follows:

$$\left(\frac{\sinh(t_j+\eta)}{\sinh(t_j-\eta)}\right)^L = \prod_{k=1;k\neq j}^{M} \frac{\sinh(t_j-t_k+2\eta)}{\sinh(t_j-t_k-2\eta)}, \qquad \text{for} \quad j = 1, 2, \ldots, R. \tag{1.4}$$

Here we have defined parameter η by $q = \exp(2\eta)$. We call such rapidities t_j *Bethe roots* at q. If Bethe roots are finite and distinct, we call them *regular*. Let $B(t)$ denote the B operator of the algebraic Bethe ansatz with rapidity t, and $|0\rangle$ the vacuum state (see e.g. [24]). It is known that the Bethe state, $B(t_1)\cdots B(t_R)|0\rangle$, gives an eigenvector of the XXZ Hamiltonian, if t_j are Bethe roots, i.e. they satisfy equations (1.4) [31]. We also call it the XXZ Bethe state. We call a nonzero Bethe state with regular Bethe roots *regular*. For a root of unity, q_0, we define η_0 by $q_0 = \exp(2\eta_0)$. We now formulate the highest weight conjecture [14, 16–18] as follows: every regular Bethe state at q_0 should be a highest weight vector of the sl_2 loop algebra.

Let us now define highest weight vectors of the sl_2 loop algebra. The generators of $U(L(sl_2))$, $x_k^{\pm}$ and h_k ($k \in \mathbf{Z}$), satisfy the defining relations:

$$\left[h_j, x_k^{\pm}\right] = \pm 2x_{j+k}^{\pm}, \qquad \left[x_j^+, x_k^-\right] = h_{j+k}, \qquad \text{for} \quad j, k \in \mathbf{Z}. \tag{1.5}$$

Here $[h_j, h_k] = 0$ and $\left[x_j^{\pm}, x_k^{\pm}\right] = 0$ for $j, k \in \mathbf{Z}$. In a representation of $U(L(sl_2))$, a vector Ω is called *a highest weight vector* if Ω is annihilated by generators x_k^+ for all integers k and such that Ω is a simultaneous eigenvector of every generator h_k ($k \in \mathbf{Z}$) [4–7, 15]:

$$x_k^+\Omega = 0, \qquad \text{for} \quad k \in \mathbf{Z}, \tag{1.6}$$

$$h_k\Omega = d_k\Omega, \qquad \text{for} \quad k \in \mathbf{Z}. \tag{1.7}$$

Here, the set of eigenvalues d_k is called the *highest weight*. We call a representation *highest weight* if it is generated by a highest weight vector. We denote it by $U\Omega$, where Ω is the highest weight vector and U denotes $U(L(sl_2))$. We assume in the paper that $U\Omega$ is finite dimensional. We can show that weight d_0 is given by a non-negative integer, which we denote by r, and also that Ω is a simultaneous eigenvector of operators $\left(x_0^+\right)^k\left(x_1^-\right)^k/(k!)^2$, i.e. $\left(x_0^+\right)^k\left(x_1^-\right)^k/(k!)^2\Omega = \lambda_k\Omega$ for $k = 1, 2, \ldots, r$. In terms of the sequence of eigenvalues λ_k: $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, we define highest weight polynomial $\mathcal{P}^{\lambda}(u)$ [13] by

$$\mathcal{P}^{\lambda}(u) = \sum_{k=0}^{r} \lambda_k(-u)^k. \tag{1.8}$$

If $U\Omega$ is irreducible, the highest weight polynomial $\mathcal{P}^{\lambda}(u)$ corresponds to the Drinfeld polynomial. It was shown that every irreducible representation is highest weight and characterized by the Drinfeld polynomial [6]. However, $U\Omega$ may be reducible. We shall show that it is indeed the case in some physical application. Here we note that a reducible representation has no Drinfeld polynomial but the highest weight polynomial.

Recently, for the XXZ spin chain at roots of unity, Fabricius and McCoy have made important observations on the highest weight conjecture [16–18]. Through the algebraic Bethe ansatz it was suggested [18] that any given XXZ Bethe state in sector A should be highest weight. Let $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ be a set of regular Bethe roots at q_0. We introduce a function $Y(v)$ as

$$Y(v) = \sum_{\ell=0}^{N-1} \frac{(\sinh(v - (2\ell+1)\eta_0))^L}{\prod_{j=1}^{R} \sinh(v - \tilde{t}_j - 2\ell\eta_0)\sinh(v - \tilde{t}_j - 2(\ell+1)\eta_0)}. \tag{1.9}$$

It follows from the Bethe ansatz equations (1.4) at q_0 that $Y(v)$ is a Laurent polynomial of variable $u = \exp(-2Nv)$ [18]. We call it the Fabricius–McCoy polynomial of the XXZ Bethe state, and denote it by $P^{\mathrm{FM}}(u)$. Furthermore, it was conjectured [18] that it should be equivalent to a 'Drinfeld polynomial' $P(u)$ through the following relation:

$$P^{\mathrm{FM}}(u) = Au^{-r/2}P(u). \tag{1.10}$$

Here A gives the normalization. However, it has not been shown whether a given XXZ Bethe state is highest weight or whether it generates an irreducible representation.

In the paper, we prove the highest weight conjecture for regular Bethe states in sectors A and B. Here sector B denotes such a sector $S^Z \equiv N/2 \pmod N$ for odd N. It is the first result of the paper. Furthermore, we discuss how far conjecture (1.10) is valid. In fact, we shall show that the Fabricius–McCoy polynomial corresponds to the highest weight polynomial, and also that there is such a regular Bethe state in sector A that generates a reducible representation. This gives the second result.

The first result is summarized as follows. Let $|R\rangle$ be a regular Bethe state at q_0 with R down-spins in sectors A or B. We have $|R\rangle = B(\tilde{t}_1)B(\tilde{t}_2)\cdots B(\tilde{t}_R)|0\rangle$ with regular Bethe roots $\tilde{t}_j$ of $|R\rangle$. By the algebraic Bethe ansatz, we shall derive the following:

$$S^{+(N)}|R\rangle = T^{+(N)}|R\rangle = 0, \tag{1.11}$$

$$(S^{+(N)})^k(T^{-(N)})^k/(k!)^2|R\rangle = \mathcal{Z}_k^+|R\rangle, \qquad \text{for} \quad k \in \mathbf{Z}_{>0}. \tag{1.12}$$

Constants $\mathcal{Z}_k^+$ will be expressed in terms of $\tilde{t}_j$, explicitly. Here we assume that a given set of regular Bethe roots at q_0 makes an isolated solution of the Bethe ansatz equations (1.4). We shall show that relations (1.11) and (1.12) are sufficient to have conditions equivalent to (1.6) and (1.7). It will thus follow that $|R\rangle$ is highest weight.

We remark that relations (1.11) generalize the $SU(2)$ symmetry of the XXX spin chain. As was shown by Takhtajan and Faddeev [32], regular XXX Bethe states are highest weight vectors of the spin $SU(2)$ symmetry. We shall discuss it in subsection 5.3.

Interestingly, novel spectral degeneracy similar to that of the XXZ spin chain at q_0 appears in the transfer matrix of the eight-vertex model at roots of unity [9, 10, 19–21], which are related to some restricted IRF models [9, 10]. Some of the degenerate eigenvectors were first discussed by Baxter [1, 2]. The elliptic degeneracy is discussed systematically by using Q matrices [19, 20]. There are some relevant researches on the sl_2 loop algebra symmetry of the XXZ spin chain at q_0 [3, 26, 27, 34]. The higher rank loop algebra symmetry has been discussed for various trigonometric-vertex models [28]. The sl_2 loop algebra and its subalgebra symmetries have been derived from the XXZ chain under twisted boundary conditions at q_0 [11, 25].

The paper consists of the following. In section 2, we make a summary of the main results. In subsection 2.1, we specify roots of unity conditions in definition 1, and we formulate a main statement on the highest weight conjecture in theorem 3. We shall discuss its proof in section 5. In subsection 2.2, for a given regular Bethe state at q_0 in sector A or B, we express the highest weight polynomial $\mathcal{P}^{\lambda}(u)$ in terms of the regular Bethe roots. We present in subsection 2.3 an irreducibility criterion of a highest weight representation. We discuss that regular Bethe states should generate Weyl modules, which may be reducible. In section 3, we introduce the algebraic Bethe ansatz and define the inhomogeneous transfer matrix of the six-vertex model. In section 4, we review the sl_2 loop algebra symmetry of the XXZ spin chain through the algebraic Bethe ansatz. In section 5, we give an outline of the proof of theorem 3. We show it for the inhomogeneous transfer matrix of the six-vertex model at roots of unity. In fact, the derivation is also valid for the homogeneous case, i.e. for the XXZ spin chain. Theorem 3 is derived from propositions 8 and 10 through lemma 6 by assuming conjecture 2. Here, propositions 8 and 10 are derived from lemmas 7 and 9, respectively. In section 6, we prove lemmas 7 and 9 explicitly. In section 7, we show that for a regular Bethe state at q_0 in sector A or B, the Fabricius–McCoy polynomial $P^{\mathrm{FM}}(u)$ (1.9) is equivalent to the highest weight polynomial. We show some examples of highest weight polynomials, and give such a regular Bethe state that generates a reducible Weyl module. In section 8, we give a concluding remark.

2. Summary of the main results

2.1. Main theorem on the highest weight conjecture

We call q *generic* if it is not a root of unity. We define a root of unity as follows.

Definition 1 (roots of unity conditions)**.** *We say that q_0 is a root of unity with $q_0^{2N} = 1$, if one of the following three conditions holds: (1) q_0 is a primitive Nth root of unity with N odd $\left(q_0^N = 1\right)$; (2) q_0 is a primitive $2N$th root of unity with N odd $\left(q_0^N = -1\right)$; (3) q_0 is a primitive $2N$th root of unity with N even $\left(q_0^N = -1\right)$. We call conditions (1) and (3) type I, and condition (2) type II.*

In the case of sector A where $S^Z \equiv 0 \pmod N$, we assume that q_0 is a root of unity with $q_0^{2N} = 1$, in the paper. In the case of sector B where $S^Z \equiv N/2 \pmod N$ with N odd, we assume that q_0 is a primitive Nth root of unity with N odd $\left(q_0^N = 1\right)$.

Here we note that the same conditions of roots of unity have been discussed in [2, 19, 20]. Let us express q_0 as $q_0 = \exp(\sqrt{-1}\pi m/N)$. In terms of m and N, roots of unity conditions (1), (2) and (3) are expressed as follows: (1) N is odd and m even; (2) N is odd and m odd; (3) N is even (m odd by definition).

We now formulate a theorem on the highest weight conjecture in the case of the inhomogeneous transfer matrix of the six-vertex model. We shall define it in section 3. Let $t_1, t_2, \ldots, t_R$ be a set of regular Bethe roots at a given value of q in the inhomogeneous case. It is known that the Bethe state, $B(t_1)\cdots B(t_R)|0\rangle$, gives an eigenvector of the inhomogeneous transfer matrix at q, if t_j satisfy the Bethe ansatz equations (3.13). Here, equations (3.13) generalize equations (1.4) in the inhomogeneous case.

Conjecture 2. *Let q_0 be a root of unity with $q_0^{2N} = 1$. Every set of regular Bethe roots at $q = q_0$ gives an isolated solution of the Bethe ansatz equations (3.13).*

Assuming conjecture 2 we shall show in section 5 the following.

Theorem 3. *Every regular Bethe state at $q = q_0$ in sectors A and B is the highest weight.*

2.2. Highest weight polynomials of regular Bethe states

Let us denote the inhomogeneous parameters by ξ_ℓ for $\ell = 1, 2, \ldots, L$, where L is the lattice size. We define functions $\phi_\xi^{\pm}(x)$ by

$$\phi_\xi^{\pm}(x) = \prod_{\ell=1}^{L} (1 - x\,\mathrm{e}^{\pm 2\xi_\ell}). \tag{2.1}$$

Recall that $t_1, t_2, \ldots, t_R$ are regular Bethe roots at a given value of q in the inhomogeneous case. We define $F^{\pm}(x)$ by

$$F^{\pm}(x) = \prod_{j=1}^{R} (1 - x\exp(\pm 2t_j)). \tag{2.2}$$

Let $|R\rangle$ be a regular Bethe state at q_0 in sector A or B in the inhomogeneous case and $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ the regular Bethe roots. At $q = q_0$, we express $F^{\pm}(x)$ as $\tilde{F}^{\pm}(x) = \prod_{j=1}^{R}(1 - x\exp(\pm 2\tilde{t}_j))$. We consider the following series with respect to x:

$$\frac{\phi_\xi^{\pm}(x)}{\tilde{F}^{\pm}(xq_0)\,\tilde{F}^{\pm}\big(xq_0^{-1}\big)} = \sum_{m=0}^{\infty} \tilde{\chi}_{\xi,m}^{\pm} x^m \qquad \text{for} \quad |x| < \min\{|\mathrm{e}^{\pm 2\tilde{t}_j}|\}. \tag{2.3}$$

Here, we define coefficients $\tilde{\chi}_{\xi,m}^{\pm}$ by power series (2.3). Explicitly, we have

$$\begin{aligned}\tilde{\chi}_{\xi,m}^{\pm} = \sum_{\rho=0}^{\min(L,m)} (-1)^{\rho} \sum_{1 \leqslant j_1 < \cdots < j_\rho \leqslant L} \exp\left(\pm \sum_{k=1}^{\rho} 2\xi_{j_k}\right) \\ \times \sum_{n_1 + \cdots + n_R = m - \rho} \mathrm{e}^{\pm \sum_{j=1}^{R} 2n_j \tilde{t}_j} \prod_{j=1}^{R} [n_j + 1]_{q_0},\end{aligned} \tag{2.4}$$

where symbol $\sum_{n_1+\cdots+n_R=m-\rho}$ denotes the sum over all non-negative integers $n_1, n_2, \ldots, n_R$ satisfying $n_1 + \cdots + n_R = m - \rho$. When $R = 0$, we set $\rho = m$.

Proposition 4. *Let q_0 be a root of unity with $q_0^{2N} = 1$. For a regular Bethe state $|R\rangle$ at q_0 in sector A or B in the inhomogeneous case such as given in theorem 3, weight d_0 is given by $r = (L - 2R)/N$, and eigenvalues λ_k of (1.8) are given by*

$$\lambda_k = \begin{cases} (-1)^{kN} \tilde{\chi}_{\xi,kN}^{+}, & \text{if} \quad q_0 \text{ is of type I}, \\ (-1)^{k(N-1)} \tilde{\chi}_{\xi,kN}^{+}, & \text{if} \quad q_0 \text{ is of type II}. \end{cases} \tag{2.5}$$

We thus obtain the highest weight polynomial $\mathcal{P}^{\lambda}(u)$ of $|R\rangle$. For type I, we have

$$\mathcal{P}^{\lambda}(u) = \begin{cases} \sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{+} u^k & \text{for} \quad \text{odd } N\big(q_0^N = 1\big) \\ \sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{+} (-u)^k & \text{for} \quad \text{even } N\big(q_0^N = -1\big), \end{cases} \tag{2.6}$$

and for type II

$$\mathcal{P}^{\lambda}(u) = \sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{+} (-u)^k \qquad \big(N : \text{odd};\ q_0^N = -1\big). \tag{2.7}$$

2.3. An irreducibility criterion of highest weight representations

In order to obtain the degenerate multiplicity of a regular Bethe state at q_0, it is fundamental to derive the dimension of a given highest weight representations [12, 13]. In fact, every finite-dimensional representation of $U(L(sl_2))$ should be given by a collection of finite-dimensional highest weight representations.

Recall that we assume in the paper that $U\Omega$ is finite dimensional. Let $\mathcal{P}^{\lambda}(u)$ be the highest weight polynomial. We introduce parameters a_k by

$$\mathcal{P}^{\lambda}(u) = \prod_{k=1}^{s}(1 - a_k u)^{m_k}. \tag{2.8}$$

Here $a_1, a_2, \ldots, a_s$ are distinct, and their multiplicities are given by $m_1, m_2, \ldots, m_s$, respectively, where $r = m_1 + \cdots + m_s$. We now introduce *highest weight parameters* $\hat{a}_i$ for $i = 1, 2, \ldots, r$, as follows:

$$\hat{a}_i = a_k \qquad \text{if} \quad m_1 + m_2 + \cdots + m_{k-1} < i \leqslant m_1 + \cdots + m_{k-1} + m_k. \tag{2.9}$$

It can be shown that $\hat{a}_j$ are nonzero [13]. We thus have three equivalent expressions for the highest weight d_k of Ω: sequence λ of eigenvalues λ_k, polynomial $\mathcal{P}^{\lambda}(u)$, and parameters $\hat{a}_j$ (i.e. parameters a_j with multiplicities m_j).

It was shown by Chari and Pressley [8] that corresponding to each irreducible finite-dimensional representation with highest weight $\mathcal{P}^{\lambda}(u)$ there exists a unique finite-dimensional highest weight module W such that any finite-dimensional highest weight module V with highest weight $\mathcal{P}^{\lambda}(u)$ is a quotient of W. The modules W are called Weyl modules [8]. Furthermore, it was shown that a Weyl module is irreducible if and only if the polynomial $\mathcal{P}^{\lambda}(u)$ has distinct roots [8]. Thus, if the highest weight parameters $\hat{a}_j$ of V are distinct, V is irreducible, and the polynomial $\mathcal{P}^{\lambda}(u)$ becomes the Drinfeld polynomial of V. We have $\dim V = 2^s$.

However, highest weight parameters $\hat{a}_j$ are not always distinct. It is shown that the highest weight representation generated by Ω, i.e. $U\Omega$, is irreducible if and only if the following holds [13]:

$$\sum_{j=0}^{s}(-1)^{s-j}\mu_{s-j}x_j^{-}\Omega = 0, \tag{2.10}$$

where μ_k $(k = 1, 2, \ldots, s)$ are given by

$$\mu_k = \sum_{1\leqslant i_1<\cdots<i_k\leqslant s} a_{i_1}\cdots a_{i_k}. \tag{2.11}$$

We note that every irreducible representation has no invariant subspace under the action of $U(L(sl_2))$ except for trivial cases. If $U\Omega$ is irreducible, the dimension is given by [4]

$$\dim U\Omega = \prod_{j=1}^{s}(m_j + 1). \tag{2.12}$$

Through criterion (2.10) we shall show such a regular Bethe state in the inhomogeneous case that generates a reducible representation (see section 7.2.3).

We shall show that for a given regular Bethe state in sector A or B, the Fabricius–McCoy polynomial (1.9) corresponds to the highest weight polynomial evaluated in (2.6) and (2.7) (see corollary 25). It thus follows that conjecture (1.10) is valid in sectors A and B if the highest weight parameters $\hat{a}_j$ are distinct. However, if $U\Omega$ is reducible, conjecture (1.10) is not valid, since it has no Drinfeld polynomial.

We have a conjecture that regular Bethe states at q_0 in sectors A and B should generate Weyl modules. A majority of regular Bethe states should have distinct highest weight parameters, while most of those with degenerate ones should generate such reducible representations that are equivalent to Weyl modules.

3. The transfer matrix of the six-vertex model

3.1. R matrix and L operator of the algebraic Bethe ansatz

In order to fix the notation, let us introduce the algebraic Bethe ansatz [24, 31, 32]. We define the R matrix of the XXZ spin chain by

$$R(z-w) = \begin{pmatrix} f(w-z) & 0 & 0 & 0 \\ 0 & g(w-z) & 1 & 0 \\ 0 & 1 & g(w-z) & 0 \\ 0 & 0 & 0 & f(w-z) \end{pmatrix}, \tag{3.1}$$

where $f(z-w)$ and $g(z-w)$ are given by

$$f(z-w) = \frac{\sinh(z-w-2\eta)}{\sinh(z-w)}, \qquad g(z-w) = \frac{\sinh(-2\eta)}{\sinh(z-w)}. \tag{3.2}$$

We recall that parameter 2η is related to q by $q = \exp(2\eta)$. We now introduce L operators for the XXZ spin chain. Let V_n be two-dimensional vector spaces for $n = 0, 1, \ldots, L$. We define an operator-valued matrix $L_n(z)$ by

$$L_n(z) = \begin{pmatrix} \sinh\left(zI_n + \eta\sigma_n^z\right) & \sinh 2\eta\sigma_n^- \\ \sinh 2\eta\sigma_n^+ & \sinh\left(zI_n - \eta\sigma_n^z\right) \end{pmatrix}. \tag{3.3}$$

Here $L_n(z)$ is a matrix acting on the auxiliary vector space V_0, where I_n and σ_n^a $(a = z, \pm)$ are operators acting on the nth vector space V_n. The symbol I denotes the two-by-two identity matrix, and $\sigma^{\pm}$ denote $\sigma^+ = E_{12}$ and $\sigma^- = E_{21}$, where they satisfy relations $E_{ij}E_{k\ell} = \delta_{j,k}E_{i\ell}$ for $i, j, k, \ell = 1, 2$. Here $\delta_{j,k}$ denotes the Kronecker delta. The symbols, $\sigma^x, \sigma^y, \sigma^z$ denote the Pauli matrices.

Let us introduce the monodromy matrix with inhomogeneous parameters ξ_n:

$$T(z; \{\xi_n\}) = L_L(z-\xi_L)\cdots L_2(z-\xi_2)L_1(z-\xi_1). \tag{3.4}$$

We call $T(z; \{\xi_n\})$ the *inhomogeneous monodromy matrix*. In terms of the R matrix and the monodromy matrices, the Yang–Baxter equation is expressed as

$$R(z-w)(T(z; \{\xi_n\}) \otimes T(w; \{\xi_n\})) = (T(w; \{\xi_n\}) \otimes T(z; \{\xi_n\}))R(z-w). \tag{3.5}$$

We express the matrix elements of the inhomogeneous monodromy matrix $T(z; \{\xi_n\})$ as

$$T(z; \{\xi_n\}) = \begin{pmatrix} A(z; \{\xi_n\}) & B(z; \{\xi_n\}) \\ C(z; \{\xi_n\}) & D(z; \{\xi_n\}) \end{pmatrix}. \tag{3.6}$$

From the Yang–Baxter equation for $T(z; \{\xi_n\})$, we have the commutation relations such as $B(w_1; \{\xi_n\})B(w_2; \{\xi_n\}) = B(w_2; \{\xi_n\})B(w_1; \{\xi_n\})$ and

$$\begin{aligned} A(w_1; \{\xi_n\})B(w_2; \{\xi_n\}) &= f(w_1-w_2)B(w_2; \{\xi_n\})A(w_1; \{\xi_n\}) \\ &\quad - g(w_1-w_2)B(w_1; \{\xi_n\})A(w_2; \{\xi_n\}). \end{aligned} \tag{3.7}$$

Here parameters w_j are arbitrary. Hereafter, we suppress the inhomogeneous parameters, ξ_n for operators A, B, C and D. We denote $B(w_j; \{\xi_n\})$ simply by $B(w_j)$.

Through the commutation relations such as (3.7) we have for $n \in \boldsymbol{Z}_{\geqslant 0}$ the following:

$$
\begin{aligned}
A(w_0)B(w_1)\cdots B(w_n) = {} & \left(\prod_{j=1}^{n} f(w_0 - w_j)\right) B(w_1)\cdots B(w_n)A(w_0) \\
& - \sum_{j=1}^{n} g(w_0 - w_j) \prod_{k\neq j}^{n} f(w_j - w_k) B(w_1)\cdots B(w_{j-1})B(w_0) \\
& \times B(w_{j+1})\cdots B(w_n)A(w_j).
\end{aligned} \tag{3.8}
$$

3.2. The inhomogeneous transfer matrix of the six-vertex model and the Bethe states

We define the *inhomogeneous transfer matrix* of the six-vertex model, $\tau_{6V}(z; \{\xi_n\})$, by

$$
\tau_{6V}(z; \{\xi_n\}) = \mathrm{tr}\, T(z; \{\xi_n\}) = A(z) + D(z). \tag{3.9}
$$

When all the inhomogeneous parameters, ξ_n, are set to be zero, we call $\tau_{6V}(z; \{\xi_n = 0\})$ the homogeneous transfer matrix of the six-vertex model and denote it simply as $\tau_{6V}(z)$. It is invariant under lattice translation. The XXZ Hamiltonian is given by the logarithmic derivative of the transfer matrix, $\tau_{6V}(z)$, at $z = \eta$:

$$
\sinh 2\eta \times \frac{\mathrm{d}}{\mathrm{d}z} \log \tau_{6V}(z)|_{z=\eta} = H_{\mathrm{XXZ}} + \frac{L}{2}\cosh 2\eta. \tag{3.10}
$$

Here we note that the XXZ coupling Δ is given by $\cosh 2\eta$.

We denote by $|0\rangle$ the vector with all spins up. We have

$$
A(z)|0\rangle = a_{\xi}^{6V}(z)|0\rangle, \qquad D(z)|0\rangle = d_{\xi}^{6V}(z)|0\rangle. \tag{3.11}
$$

Here $a_{\xi}^{6V}(z)$ and $d_{\xi}^{6V}(z)$ are given by

$$
a_{\xi}^{6V}(z) = \prod_{n=1}^{L} \sinh(z - \xi_n + \eta), \qquad d_{\xi}^{6V}(z) = \prod_{n=1}^{L} \sinh(z - \xi_n - \eta). \tag{3.12}
$$

It is shown [31] that the vector $B(t_1)B(t_2)\cdots B(t_R)|0\rangle$ is an eigenvector of the inhomogeneous transfer matrix $\tau_{6V}(z; \{\xi_n\})$ if rapidities $t_1, t_2, \ldots, t_R$ satisfy the Bethe ansatz equations

$$
\frac{a_{\xi}^{6V}(t_j)}{d_{\xi}^{6V}(t_j)} = \prod_{k=1, k\neq j}^{R} \frac{f(t_k - t_j)}{f(t_j - t_k)}, \qquad \text{for} \quad j = 1, 2, \ldots, R. \tag{3.13}
$$

In the same way as the homogeneous case for (1.4), we call the eigenvector $B(t_1)B(t_2)\cdots B(t_R)|0\rangle$ *the Bethe ansatz eigenvector* or *the Bethe state*, briefly. Here we call $t_1, t_2, \ldots, t_R$, the *Bethe roots*, and call them regular if they are finite and distinct. Furthermore, we call the Bethe state *regular*, if it is nonzero and the Bethe roots are regular.

4. The sl_2 loop algebra symmetry at roots of unity

4.1. Generators of the quantum groups

The quantum affine algebra $U_q(\hat{sl_2})$ is an associative algebra over $\mathbf{C}$ generated by $e_i^{\pm}$, $K_i^{\pm}$ for $i = 0, 1$ with the following relations:

$$
\begin{aligned}
& K_i K_i^{-1} = K_i K_i^{-1} = 1, \qquad K_i e_i^{\pm} K_i^{-1} = q^{\pm 2} e_i^{\pm}, \qquad K_i e_j^{\pm} K_i^{-1} = q^{\mp 2} e_j^{\pm} \quad (i \neq j), \\
& \left[e_i^{+}, e_j^{-}\right] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\
& \left(e_i^{\pm}\right)^3 e_j^{\pm} - [3]_q \left(e_i^{\pm}\right)^2 e_j^{\pm} e_i^{\pm} + [3]_q e_i^{\pm} e_j^{\pm} \left(e_i^{\pm}\right)^2 - e_j^{\pm} \left(e_i^{\pm}\right)^3 = 0 \qquad (i \neq j).
\end{aligned} \tag{4.1}
$$

Here q is generic and $[n]_q$ denotes the q-integer of an integer n: $[n]_q = (q^n - q^{-n})/(q - q^{-1})$. The algebra $U_q(\hat{sl_2})$ is also a Hopf algebra over $\mathbf{C}$ with co-multiplication

$$\begin{aligned} &\Delta\big(e_i^+\big) = e_i^+ \otimes K_i + 1 \otimes e_i^+, \qquad \Delta\big(e_i^-\big) = e_i^- \otimes 1 + K_i^{-1} \otimes e_i^-, \\ &\Delta(K_i) = K_i \otimes K_i, \end{aligned} \tag{4.2}$$

and antipode $S(K_i) = K_i^{-1}$, $S(e_i) = -e_i^+ K_i^{-1}$, $S(e_i^-) = -K_i e_i^-$.

The quantum algebra $U_q(sl_2)$ is an associative Hopf algebra over $\mathbf{C}$ generated by elements $e^{\pm}$ and K with the same defining relations of the Hopf algebra as those for $e_1^{\pm}$ and K_1 of $U_q(\hat{sl_2})$.

We now introduce evaluation representations for $U_q(\hat{sl_2})$ [22]. For a given nonzero complex number a there is a homomorphism of algebras φ_a: $U_q(\hat{sl_2}) \to U_q(sl_2)$ such that $\varphi_a\big(e_0^{\pm}\big) = q^{\mp 1} a^{\pm 1} \mathrm{e}^{\mp}$, $\varphi_a\big(e_1^{\pm}\big) = \mathrm{e}^{\pm}$, $\varphi_a(K_0) = K^{-1}$ and $\varphi_a(K_1) = K$. Let us denote by (π, V) a representation of an algebra $\mathcal{A}$ such that $\pi(x)$ give linear maps on vector space V for $x \in \mathcal{A}$. For a given finite-dimensional representation (π_V, V) of $U_q(sl_2)$, we have a finite-dimensional representation $(\pi_{V(a)}, V(a))$ of $U_q(\hat{sl_2})$ through homomorphism φ_a, i.e. $\pi_{V(a)}(x) = \pi_V(\varphi_a(x))$ for $x \in U_q(\hat{sl_2})$. We call $(\pi_{V(a)}, V(a))$ or $V(a)$ the *evaluation representation* of V and nonzero parameter a the *evaluation parameter* of $V(a)$.

We consider such finite-dimensional representations of $U_q(\hat{sl_2})$, where K_i are equivalent to q^{H_i} with diagonal matrices H_i for $i = 0, 1$. In the representations, we denote by $K_i^{1/2}$ the square root of K_i. We introduce the following operators for $i = 0, 1$:

$$\hat{e}_i^+ = q^{n_i} K_i^{-1/2} e_i^+, \qquad \hat{e}_i^- = q^{-n_i} e_i^- K_i^{1/2}. \tag{4.3}$$

Here n_i $(i = 0, 1)$ are arbitrary. The operators $\hat{e}_i^{\pm}$ satisfy the same defining relations (4.1) with $e_i^{\pm}$ and the following co-multiplication [23]:

$$\Delta\big(\hat{e}_i^{\pm}\big) = \hat{e} \otimes K_i^{1/2} + K_i^{-1/2} \otimes \hat{e}_i^{\pm}. \tag{4.4}$$

Taking the co-multiplication $L - 1$ times we have

$$\Delta^{(L-1)}\big(\hat{e}_i^{\pm}\big) = \sum_{j=1}^{L} \big(K_i^{-1/2}\big)^{\otimes(j-1)} \otimes \hat{e}_i^{\pm} \otimes \big(K_i^{1/2}\big)^{\otimes(L-j)}. \tag{4.5}$$

Let us denote by (π_1, V_1) such a two-dimensional irreducible representation of $U_q(sl_2)$ where generators $\mathrm{e}^{\pm}$ and K are represented by the Pauli matrices $\sigma^{\pm}$ and q^{σ^Z}, respectively, i.e. we have $\pi_1(\mathrm{e}^{\pm}) = \sigma^{\pm}$ and $\pi_1(K) = q^{\sigma^Z}$. For the evaluation representation $V_1(a)$, we take the Lth tensor product, $V_1(a)^{\otimes L}$. Setting $a = q$, we denote by $S^{\pm}$ and $T^{\pm}$ the matrix representations of generators $\hat{e}_0^{\mp}$ and $\hat{e}_1^{\pm}$ acting on $V_1(q)^{\otimes L}$, respectively. Here we note $S^{\pm} = \pi_{V_1(q)} \otimes \cdots \otimes \pi_{V_1(q)}\big(\Delta^{(L-1)}\big(\hat{e}_0^{\mp}\big)\big)$. Explicitly, we have

$$\begin{aligned} S^{\pm} &= \sum_{j=1}^{L} q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2}, \\ T^{\pm} &= \sum_{j=1}^{L} q^{-\sigma^Z/2} \otimes \cdots \otimes q^{-\sigma^Z/2} \otimes \sigma_j^{\pm} \otimes q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2}. \end{aligned} \tag{4.6}$$

Here we have set $n_0 = n_1 = 1/2$. The symbol $\sigma_j^{\pm}$ denotes the Pauli matrices $\sigma^{\pm}$ acting on the jth component of the tensor product. We denote by q^{S^Z} the matrix representation of $K_1^{1/2}$ acting on the tensor product $V_1(q)^{\otimes L}$. We have

$$q^{S^Z} = \big(q^{\sigma^Z/2}\big)^{\otimes L} = q^{\sigma^Z/2} \otimes \cdots \otimes q^{\sigma^Z/2}. \tag{4.7}$$

We recall that S^Z denotes the Z-component of the total spin operator, $S^Z = \sum_{j=1}^{L} \sigma_j^Z/2$.

We define the q-factorial of n by $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$. Here, q is generic and not a root of unity. We introduce the following notation for the nth power of an operator X divided by the q-factorial of n:

$$(X)_q^{(n)} = (X)^n/[n]_q!. \tag{4.8}$$

It is easy to show the following [14]:

$$(T^{\pm})_q^{(n)} = \sum_{1\leqslant j_1<\cdots<j_n\leqslant L} q^{-\frac{n}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{n}{2}\sigma^Z} \otimes \sigma_{j_1}^{\pm} \otimes q^{-\frac{(n-2)}{2}\sigma^Z} \otimes \cdots \otimes q^{-\frac{(n-2)}{2}\sigma^Z} \otimes \sigma_{j_2}^{\pm} \otimes q^{-\frac{(n-4)}{2}\sigma^Z} \otimes \cdots \otimes \sigma_{j_N}^{\pm} \otimes q^{\frac{n}{2}\sigma^Z} \otimes \cdots \otimes q^{\frac{n}{2}\sigma^Z}. \tag{4.9}$$

We derive operators $S^{\pm(N)}$ and $T^{\pm(N)}$ defined by (1.2) of section 1 through the following limit:

$$S^{\pm(N)} = \lim_{q\to q_0} (S^{\pm})_q^{(N)}, \qquad T^{\pm(N)} = \lim_{q\to q_0} (T^{\pm})_q^{(N)}. \tag{4.10}$$

Here we recall that q_0 denotes a root of unity satisfying $q_0^{2N} = 1$.

4.2. *Quantum group generators through infinite rapidities*

Let us define function $n_\xi(z)$ by

$$n_\xi(z) = \prod_{j=1}^{L} \sinh(z - \xi_j), \tag{4.11}$$

and function $\hat{g}(z)$ by

$$\hat{g}(z) = \begin{cases} 2\exp(-z)\sinh 2\eta & (\mathrm{Re}\, z > 0), \\ -2\exp(z)\sinh 2\eta & (\mathrm{Re}\, z < 0). \end{cases} \tag{4.12}$$

We normalize operators $A(z), \ldots, D(z)$ as follows:

$$\begin{aligned} \hat{A}(z) &= A(z)/n_\xi(z), & \hat{B}(z) &= B(z)/(\hat{g}(z)n_\xi(z)), \\ \hat{D}(z) &= D(z)/n_\xi(z), & \hat{C}(z) &= C(z)/(\hat{g}(z)n_\xi(z)). \end{aligned} \tag{4.13}$$

Taking the limit of infinite rapidities for the inhomogeneous case, we have

$$\begin{aligned} \hat{A}(\pm\infty) &= q^{\pm S^Z}, & \hat{B}(\infty) &= V^- T^- V^+, & \hat{B}(-\infty) &= V^+ S^- V^-, \\ \hat{D}(\pm\infty) &= q^{\mp S^Z}, & \hat{C}(\infty) &= V^+ S^+ V^-, & \hat{C}(-\infty) &= V^- T^+ V^+, \end{aligned} \tag{4.14}$$

where $V^{\pm}$ are given by the following diagonal matrices [11]:

$$(V^{\pm})_{k_1,\ldots,k_L}^{j_1,\ldots,j_L} = \exp\left(\pm \sum_{i=1}^{L} \xi_i j_i\right) \delta_{j_1,k_1} \cdots \delta_{j_L,k_L}, \tag{4.15}$$

where $j_\ell, k_\ell = 1, 2$ for $\ell = 1, 2, \ldots, L$. Let us define operators $S_\xi^{\pm}$ and $T_\xi^{\pm}$ by

$$S_\xi^{\pm} = V^+ S^{\pm} V^-, \qquad T_\xi^{\pm} = V^- T^{\pm} V^+. \tag{4.16}$$

We can show that $S_\xi^{\pm}, T_\xi^{\pm}$ and q^{S^Z} satisfy the defining relations of $U_q(L(sl_2))$ through the evaluation homomorphism [22]. Recall generators $\hat{e}_i^{\pm}$ for $i = 0, 1$ defined by (4.3) with $n_0 = n_1 = 1/2$. In the tensor product $\otimes_{j=1}^{L} V_1(q_j) = V_1(q\,\mathrm{e}^{2\xi_1}) \otimes V_1(q\,\mathrm{e}^{2\xi_2}) \otimes \cdots \otimes V_1(q\,\mathrm{e}^{2\xi_L})$, generators $\hat{e}_i^{\pm}$ for $i = 0, 1$ are related to $S_\xi^{\pm}$ and $T_\xi^{\pm}$ as follows:

$$S_\xi^{\pm} = V^- \otimes_{j=1}^{L} \pi_{V_1(q_j)} \Delta^{(L-1)}\left(\hat{e}_0^{\mp}\right) V^+, \qquad T_\xi^{\pm} = V^- \otimes_{j=1}^{L} \pi_{V_1(q_j)} \Delta^{(L-1)}\left(\hat{e}_1^{\pm}\right) V^+.$$

Here $q_j = q \exp 2\xi_j$ for $j = 1, 2, \ldots, L$. Thus, $S_\xi^{\pm}$ and $T_\xi^{\pm}$ satisfy the same defining relations of $U_q(L(sl_2))$ as generators $\hat{e}_0^{\mp}$ and $\hat{e}_1^{\pm}$, respectively.

4.3. Complete N-strings

Let N be a positive integer.

Definition 5 (complete N-string)**.** *We call a set of rapidities z_j a complete N-string, if they have the following relation:*

$$z_j = \Lambda + \eta(N + 1 - 2j) \qquad (j = 1, 2, \dots, N). \tag{4.17}$$

We call the parameter Λ the centre of the N-string.

Setting $w_j = z_j$ with z_j being the complete N-string, we have the following:

$$\prod_{k=1,k\neq j}^{N} f(z_j - z_k) = \begin{cases} 0 & (j \neq N) \\ [N]_q & (j = N), \end{cases} \qquad \prod_{k=1,k\neq j}^{N} f(z_k - z_j) = \begin{cases} [N]_q & (j = 1) \\ 0 & (j \neq 1). \end{cases} \tag{4.18}$$

Applying (4.18) into (3.8) with $n = N$, and sending the centre Λ of the complete N-string (4.17) to infinity, we have

$$A(w_0)\left(T_\xi^-\right)^N = q^N \left(T_\xi^-\right)^N A(w_0) - [N]_q B(w_0)\left(T_\xi^-\right)^{N-1} q^{S^Z} \mathrm{e}^{w_0}. \tag{4.19}$$

Similarly, we have

$$D(w_0)\left(T_\xi^-\right)^N = q^{-N}\left(T_\xi^-\right)^N D(w_0) - [N]_q B(w_0)\left(T_\xi^-\right)^{N-1} q^{-S^Z}(-\mathrm{e}^{w_0}). \tag{4.20}$$

We thus have

$$\begin{aligned}(A(w_0) + D(w_0))\left(T_\xi^-\right)_q^{(N)} &= \left(T_\xi^-\right)_q^{(N)}(q^N A(w_0) + q^{-N} D(w_0)) \\ &\quad - \mathrm{e}^{w_0} B(w_0)\left(T_\xi^-\right)_q^{(N-1)}\left(q^{S^Z} - q^{-S^Z}\right).\end{aligned} \tag{4.21}$$

Taking the limit $\Lambda \to -\infty$, we have the commutation relation for S_ξ^-. Similarly, we derive commutation relations for $\left(S_\xi^+\right)^N$ and $\left(T_\xi^+\right)^N$. In summary we have the following:

$$\begin{aligned}(A(w_0) + D(w_0))\left(S_\xi^\pm\right)_q^{(N)} &= \left(S_\xi^\pm\right)_q^{(N)}(q^{-N} A(w_0) + q^N D(w_0)) \\ &\quad + \mathrm{e}^{\pm w_0} X(w_0)\left(S_\xi^\pm\right)_q^{(N-1)}\left(q^{S^Z} - q^{-S^Z}\right),\end{aligned} \tag{4.22}$$

$$\begin{aligned}(A(w_0) + D(w_0))\left(T_\xi^\pm\right)_q^{(N)} &= \left(T_\xi^\pm\right)_q^{(N)}(q^N A(w_0) + q^{-N} D(w_0)) \\ &\quad - \mathrm{e}^{\mp w_0} X(w_0)\left(T_\xi^\pm\right)_q^{(N-1)}\left(q^{S^Z} - q^{-S^Z}\right),\end{aligned} \tag{4.23}$$

where $X(w_0) = C(w_0)$ for S_ξ^+ and T_ξ^+, while $X(w_0) = B(w_0)$ for S_ξ^- and T_ξ^-.

4.4. $S_\xi^{\pm(N)}$ and $T_\xi^{\pm(N)}$ as generators of the sl_2 loop algebra

Let us recall that in sector A where $S^Z \equiv 0 \pmod N$ q_0 is a root of unity with $q_0^{2N} = 1$, (cf definition 1), while in sector B where $S^Z \equiv N/2 \pmod N$ with N odd, q_0 is a primitive Nth root of unity.

It follows from (4.22) and (4.23) that $S_\xi^{\pm(N)}$ and $T_\xi^{\pm(N)}$ (anti-)commute with the transfer matrix of the six-vertex model $\tau_{6V}(z; \{\xi_n\})$ in the cases of sector A and B:

$$\begin{aligned} S_\xi^{\pm(N)} \tau_{6V}(z; \{\xi_n\}) &= q_0^N \tau_{6V}(z; \{\xi_n\}) S_\xi^{\pm(N)}, \\ T_\xi^{\pm(N)} \tau_{6V}(z; \{\xi_n\}) &= q_0^N \tau_{6V}(z; \{\xi_n\}) T_\xi^{\pm(N)}. \end{aligned} \tag{4.24}$$

It is readily derived from the (anti-)commutation relations (4.24) that the operators $S^{\pm(N)}$ and $T^{\pm(N)}$ commute with the XXZ Hamiltonian in the cases of sectors A and B. Here we recall that the XXZ Hamiltonian H_{XXZ} is given by the logarithmic derivative of the homogeneous transfer matrix $\tau_{6V}(z)$.

We now show that $S_\xi^{\pm(N)}$ and $T_\xi^{\pm(N)}$ generate the sl_2 loop algebra [14]. When q_0 is of type I, we set

$$\begin{aligned} &E_0^+ = T_\xi^{-(N)}, \qquad E_0^- = T_\xi^{+(N)}, \qquad E_1^+ = S_\xi^{+(N)}, \qquad E_1^- = S_\xi^{-(N)}, \\ &-H_0 = H_1 = \frac{2}{N} S^Z. \end{aligned} \tag{4.25}$$

When q_0 is of type II, we set

$$\begin{aligned} &E_0^+ = \sqrt{-1} T_\xi^{-(N)}, \qquad E_0^- = \sqrt{-1} T_\xi^{+(N)}, \qquad E_1^+ = \sqrt{-1} S_\xi^{+(N)}, \\ &E_1^- = \sqrt{-1} S_\xi^{-(N)}, \qquad -H_0 = H_1 = \frac{2}{N} S^Z. \end{aligned} \tag{4.26}$$

Here $\sqrt{-1}$ denotes the square root of -1. (See also (A.13) of [9].) Then, operators $E_j^{\pm}$, H_j for $j = 0, 1$, satisfy the defining relations of the sl_2 loop algebra $U(L(sl_2))$ [14]:

$$H_0 + H_1 = 0, \qquad \left[H_i, E_j^{\pm}\right] = \pm a_{ij} E_j^{\pm}, \qquad (i, j = 0, 1) \tag{4.27}$$

$$\left[E_i^+, E_j^-\right] = \delta_{ij} H_j, \qquad (i, j = 0, 1) \tag{4.28}$$

$$\left[E_i^{\pm}, \left[E_i^{\pm}, \left[E_i^{\pm}, E_j^{\pm}\right]\right]\right] = 0, \qquad (i, j = 0, 1, i \neq j). \tag{4.29}$$

Here, the Cartan matrix (a_{ij}) of $A_1^{(1)}$ is defined by

$$\begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{4.30}$$

We obtain relations (4.27), (4.28) and (4.29) from the fact that $S_\xi^{\pm}$ and $T_\xi^{\pm}$ are generators of the quantum group $U_q(\hat{sl}_2)$. The Serre relations (4.29) hold if q_0 is a primitive $2N$th root of unity, or a primitive Nth root of unity with N odd [14]. We derive it through the higher order quantum Serre relations due to Lusztig [30]. The Cartan relations (4.27) hold for generic q. Relation (4.28) holds for the identification (4.25) when q_0 is a root of unity of type I, and for the identification (4.26) when q_0 is a root of unity of type II. In the case of sector A ($S^Z \equiv 0$ (mod N) and q_0 is a root of unity with $q_0^{2N} = 1$), we have the commutation relation [14]

$$\left[S_\xi^{+(N)}, S_\xi^{-(N)}\right] = (-1)^{N-1} q^N \frac{2}{N} S^Z. \tag{4.31}$$

Here the sign factor $(-1)^{N-1} q^N$ is given by 1 or -1 when q is a root of unity of type I or II, respectively. In the case of sector B ($S^Z \equiv N/2$ (mod N) with N odd and q_0 a primitive Nth root of unity), we have the commutation relation

$$\left[S_\xi^{+(N)}, S_\xi^{-(N)}\right] = \frac{2}{N} S^Z. \tag{4.32}$$

Using the auto-morphism of the loop algebra $\theta\left(E_0^{\pm}\right) = E_1^{\pm}$ and $\theta(H_0) = H_1$, we derive another identification. For instance, for the type I case, we may set

$$\begin{aligned} &E_0^+ = S_\xi^{+(N)}, \qquad E_0^- = S_\xi^{-(N)}, \qquad E_1^+ = T_\xi^{-(N)}, \qquad E_1^- = T_\xi^{+(N)}, \\ &H_0 = -H_1 = \frac{2}{N} S^Z. \end{aligned} \tag{4.33}$$

The identification (4.33) with $\xi_n = 0$ for all n is given in [14].

Generators $x_k^{\pm}$ and h_k for $k \in \boldsymbol{Z}$ satisfying the defining relations (1.5) are the classical limits of the Drinfeld generators [6, 15]. However, we also call them Drinfeld generators for simplicity. There is an isomorphism between the Drinfeld generators and the Chevalley generators as follows [6, 15]:

$$E_1^{\pm} \mapsto x_0^{\pm}, \qquad E_0^{+} \mapsto x_1^{-}, \qquad E_0^{-} \mapsto x_{-1}^{+}, \qquad -H_0 = H_1 \mapsto h_0. \tag{4.34}$$

For roots of unity of type I, i.e. q_0 is a primitive Nth root of unity with N odd $\left(q_0^N = 1\right)$ or q is a $2N$th primitive root of unity with N even $\left(q_0^N = -1\right)$, through isomorphism (4.34) and identification (4.25) we have the following correspondence:

$$\begin{aligned} &x_0^{+} = S_{\xi}^{+(N)}, \qquad x_0^{-} = S_{\xi}^{-(N)}, \qquad x_{-1}^{+} = T_{\xi}^{+(N)}, \qquad x_1^{-} = T_{\xi}^{-(N)}, \\ &h_0 = \frac{2}{N} S^Z. \end{aligned} \tag{4.35}$$

Here relations (4.35) are valid both in the sector $S^Z \equiv 0 \pmod N$ and in the sector $S^Z \equiv N/2 \pmod N$. For roots of unity of type II, i.e. when q_0 is a $2N$th primitive root of unity with N odd $\left(q_0^N = -1\right)$, through the isomorphism (4.34) and the identification (4.26), we have the following:

$$\begin{aligned} &x_0^{+} = \sqrt{-1} S_{\xi}^{+(N)}, \qquad x_0^{-} = \sqrt{-1} S_{\xi}^{-(N)}, \qquad x_{-1}^{+} = \sqrt{-1} T_{\xi}^{+(N)}, \\ &x_1^{-} = \sqrt{-1} T_{\xi}^{-(N)}, \qquad h_0 = \frac{2}{N} S^Z. \end{aligned} \tag{4.36}$$

Let the symbol $U_q^{\text{res}}(g)$ denote the algebra generated by the q-divided powers of the Chevalley generators of a Lie algebra g such as $\left(e_j^{\pm}\right)_q^{(N)}$ [7]. The correspondence of the algebra $U_{q_0}^{\text{res}}(g)$ at a root of unity, q_0, to the Lie algebra $U(g)$ was obtained essentially through the machinery introduced by Lusztig [29, 30] both for finite-dimensional simple Lie algebras and infinite-dimensional affine Lie algebras. In fact, by using the higher order quantum Serre relations [30], it has been shown that the affine Lie algebra $U(\hat{sl}_2)$ is generated by $\left(e_j^{\pm}\right)_{q_0}^{(N)}$ at roots of unity. However, in the case of the affine Lie algebras $\hat{g}$, the highest weight conditions for the Drinfeld generators are different from those for the Chevalley generators. Through the highest weight vectors of the Drinfeld generators, finite-dimensional representations were discussed by Chari and Pressley for $U_{q_0}^{\text{res}}(\hat{g})$ [7].

5. The outline of the proof of the highest weight conjecture

5.1. Sufficient conditions of a highest weight vector

Lemma 6. *Suppose that $x_0^{\pm}, x_{-1}^{+}, x_1^{-}$ and h_0 satisfy the defining relations of $U(L(sl_2))$, and $x_k^{\pm}$ and h_k ($k \in \boldsymbol{Z}$) are generated from them. If a vector $|\Phi\rangle$ satisfies the following:*

$$x_0^{+}|\Phi\rangle = x_{-1}^{+}|\Phi\rangle = 0, \tag{5.1}$$

$$h_0|\Phi\rangle = r|\Phi\rangle, \tag{5.2}$$

$$\left(x_0^{+}\right)^{(n)}\left(x_1^{-}\right)^{(n)}|\Phi\rangle = \lambda_n|\Phi\rangle \qquad \text{for} \quad n = 1, 2, \ldots, r, \tag{5.3}$$

where r is a non-negative integer and λ_n are complex numbers. Here $(x)^{(n)}$ denotes $x^n/n!$. Then $|\Phi\rangle$ is highest weight, i.e. we have

$$x_k^{+}|\Phi\rangle = 0 \qquad (k \in \boldsymbol{Z}), \tag{5.4}$$

$$h_k|\Phi\rangle = d_k|\Phi\rangle \qquad (k \in \boldsymbol{Z}), \tag{5.5}$$

where d_k are complex numbers.

Lemma 6 will be shown in appendix A. Here we note that conditions (5.1), (5.2) and (5.3) are also necessary for $|\Phi\rangle$ to be highest weight.

We shall derive theorem 3 on the highest weight conjecture through lemma 6. Conditions (5.1) and (5.3) correspond to (1.11) and (1.12), respectively. We call conditions (5.1) and (5.3) *annihilation property* and *diagonal property*, respectively. We note that conditions (5.4) and (5.5) correspond to (1.6) and (1.7), respectively.

5.2. Isolated solutions of the Bethe ansatz equations

As far as the Bethe ansatz equations are concerned, every isolated solution at a root of unity, q_0, is continuously extended to a solution at generic q near q_0. Note that the Bethe ansatz equations (1.4) (and (3.13)) are expressed in terms of rational functions of q of finite degree.

We now assume conjecture 2. Let $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ be a set of regular Bethe roots at q_0 forming an isolated solution of Bethe ansatz equations. It follows from conjecture 2 that there exist such regular Bethe roots at $q, t_1, t_2, \ldots, t_R$, that approach the regular Bethe roots at $q_0, \tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$, respectively, when q goes to q_0. The regular Bethe state $|R\rangle$ at q_0 associated with $\tilde{t}_j$ is thus given by

$$|R\rangle = B(\tilde{t}_1, \eta_0)B(\tilde{t}_2, \eta_0)\cdots B(\tilde{t}_R, \eta_0)|0\rangle. \tag{5.6}$$

Here we recall $q = \exp(2\eta)$, and the η-dependence has been explicitly expressed as $B(w, \eta)$. Let us denote by $|R\rangle_q$ the regular Bethe state at q

$$|R\rangle_q = B(t_1, \eta)B(t_2, \eta)\cdots B(t_R, \eta)|0\rangle. \tag{5.7}$$

Then, we have the following:

$$|R\rangle = \lim_{q\to q_0} |R\rangle_q. \tag{5.8}$$

We remark that conjecture 2 is supported by an extensive study of numerical solutions of the Bethe ansatz equations near roots of unity [16]. For $R = 1$, we can show it explicitly with analytic expressions of Bethe roots in terms of q.

5.3. Annihilation property

Let us discuss the derivation of relations (1.11) (i.e. (5.1)). We construct operators $S_\xi^{+(N)}$ and $T_\xi^{+(N)}$ as follows:

$$S_\xi^{+(N)} = \lim_{q\to q_0} \frac{1}{[N]_q!}(\hat{C}(\infty, \eta))^N, \qquad T_\xi^{+(N)} = \lim_{q\to q_0} \frac{1}{[N]_q!}(\hat{C}(-\infty, \eta))^N. \tag{5.9}$$

We evaluate the action of the operator $S_\xi^{+(N)}$ on the vector $|R\rangle$ by the limiting procedure

$$S_\xi^{+(N)}|R\rangle = \lim_{q\to q_0} \left\{ \frac{1}{[N]_q!}(\hat{C}(\infty, \eta))^N B(t_1, \eta)\cdots B(t_R, \eta)|0\rangle \right\}. \tag{5.10}$$

Let us recall that L is the lattice size and R is the number of regular Bethe roots. We denote by $\Sigma_R = \{1, 2, \ldots, R\}$ the set of indices of the regular Bethe roots, $t_1, t_2, \ldots, t_R$. For a given set S we denote by $|S|$ the number of elements. We express by $\sum_{A\subset B}^{|A|=n}$ the sum over all such subsets A of B that have n elements. The following lemma will be shown in section 6.

Lemma 7. *Let $t_1, t_2, \ldots, t_R$ be regular Bethe roots at generic q. For a given positive integer N_c, we have*

$$\frac{1}{[N_c]_q!}(\hat{C}(\pm\infty))^{N_c} B(t_1)B(t_2)\cdots B(t_R)|0\rangle = \sum_{S_{N_c}\subseteq\Sigma_R}^{|S_{N_c}|=N_c} \prod_{\ell\in\Sigma_R\setminus S_{N_c}} B(t_\ell)|0\rangle$$
$$\times \exp\left(\pm\sum_{j\in S_{N_c}} t_j\right)\prod_{j\in S_{N_c}}\left(a_\xi^{6V}(t_j)\prod_{k\in\Sigma_R\setminus S_{N_c}} f(t_j-t_k)\right)$$
$$\times(-1)^{N_c}(q^{\pm1}-q^{\mp1})^{N_c}\times\prod_{\ell=0}^{N_c-1}\left[\frac{L}{2}-R+N_c-\ell\right]_q. \tag{5.11}$$

Assuming conjecture 2 we have the following:

Proposition 8. *(i) When L is even and q_0 a root of unity with $q_0^{2N}=1$, every regular Bethe state $|R\rangle$ at q_0 is annihilated by operators $S^{+(N)}$ and $T^{+(N)}$ in any sector of $S^Z\in\mathbf{Z}$:*

$$S_\xi^{+(N)}|R\rangle = T_\xi^{+(N)}|R\rangle = 0. \tag{5.12}$$

(ii) When L is odd, N is odd and q_0 a primitive Nth root of unity, every regular Bethe state $|R\rangle$ at q_0 is annihilated by operators $S_\xi^{+(N)}$ and $T_\xi^{+(N)}$ in any sector of S^Z, where S^Z takes half-integers.

Proof. Let us put $N_c = N$ in (5.11). In the case of even L, the product: $\prod_{\ell=0}^{N-1}[L/2-R+N-\ell]_{q_0}$ vanishes if $q_0^{2N}=1$. Thus, by taking the limit $q\to q_0$, it follows from (5.11) that operators $S^{+(N)}$ and $T^{+(N)}$ annihilate the regular Bethe state $|R\rangle$. In the case of odd L, the product: $\prod_{\ell=0}^{N-1}[L/2-R+N-\ell]_{q_0}$ vanishes if $q_0^N=1$. It follows that operators $S^{+(N)}$ and $T^{+(N)}$ annihilate the regular Bethe state $|R\rangle$. □

We note that when $q=\pm1$ and $N=1$, expression (5.11) leads to another proof for the spin $SU(2)$ invariance of the XXX Bethe states shown in [32]. In fact, the right hand side of (5.11) vanishes when $q=\pm1$ and $N=1$, since factor $q-q^{-1}$ vanishes.

5.4. Diagonal property

Let us consider the limiting procedure of sending rapidities z_j and z_k to infinity in the products of operators such as $\hat{B}(z_j)$ and $\hat{C}(z_k)$. Since they are matrices of finite sizes, the infinite limiting procedure does not depend on the order of sending arguments z_j to infinity. For instance, we have

$$\lim_{z_1\to\infty}\big(\lim_{z_2\to\infty}\hat{C}(z_1)\hat{B}(z_2)\big) = \big(\lim_{z_1\to\infty}\hat{C}(z_1)\big)\big(\lim_{z_2\to\infty}\hat{B}(z_2)\big). \tag{5.13}$$

Each of the matrix elements of operators $\hat{B}(z_j)$ and $\hat{C}(z_j)$ is written as a sum of products of 2×2 matrices such as $\sinh\big(z_j\pm\eta\sigma_n^z\big)/\sinh z_j$ and $\sinh 2\eta\sigma_n^\pm/\sinh z_j$. Here we recall normalization (4.13).

In addition to regular Bethe roots at generic $q, t_1, t_2, \ldots, t_R$, we introduce kN rapidities, $z_1, z_2, \ldots, z_{kN}$, forming a complete kN-string: $z_j = \Lambda+(kN+1-2j)\eta$ for $j=1,2,\ldots,kN$. Here we recall definition (4.17) of the complete N-string. We calculate the action of $\big(S_\xi^{+(N)}\big)^k\big(T_\xi^{-(N)}\big)^k$ on the Bethe state at q_0, $|R\rangle = B(\tilde{t}_1)\cdots B(\tilde{t}_R)|0\rangle$, as follows:

$$\big(S_\xi^{+(N)}\big)^k\big(T_\xi^{-(N)}\big)^k|R\rangle = \lim_{q\to q_0}\left(\lim_{\Lambda\to\infty}\frac{1}{([N]_q!)^k}(\hat{C}(\infty))^{kN}\right.$$
$$\left.\times\frac{1}{([N]_q!)^k}\hat{B}(z_1,\eta)\cdots\hat{B}(z_{kN},\eta_n)B(t_1,\eta)\cdots B(t_R,\eta)|0\rangle\right). \tag{5.14}$$

Associated with the limit $\Lambda \to \pm\infty$, we define $\epsilon_\nu^\pm$ by

$$\epsilon_\nu^\pm = \exp(\mp 2z_\nu) = \epsilon_0^\pm q^{\pm 2\nu}, \qquad \text{for} \quad \nu = 0, 1, \ldots, kN. \tag{5.15}$$

Here $\epsilon_0^\pm = \exp(\mp\Lambda \mp (kN+1)\eta)$. We expand $\hat{B}(z_\nu)$ at infinities:

$$\hat{B}(z_\nu) = \sum_{n=0}^{\infty} \hat{b}_{\xi,n}^{\pm} \left(\epsilon_\nu^\pm\right)^n \qquad (\Lambda \to \pm\infty). \tag{5.16}$$

Infinite series (5.16) is convergent if $\epsilon_\nu^\pm$ is small enough. The matrix $\hat{B}(z_\nu)$ has a finite number of matrix elements which are given by sums of products of 2×2 matrices such as $\sinh\left(z_j - \xi_n \pm \eta\sigma_n^z\right)/\sinh(z_j - \xi_n)$ and $\sinh 2\eta\sigma_n^\pm/\sinh(z_j - \xi_n)$.

Let $t_1, t_2, \ldots, t_R$ be a set of regular Bethe roots at generic q. We define $s_j^\pm(x)$ by

$$s_j^\pm(x) = 1 - x\exp(\pm 2t_j) \qquad \text{for} \quad j = 1, 2, \ldots, R. \tag{5.17}$$

For a subset J of Σ_R, i.e. $J \subset \Sigma_R$, we define $F_J^\pm(x)$ by

$$F_J^\pm(x) = \prod_{\ell \in \Sigma_R \setminus J} s_j^\pm(x). \tag{5.18}$$

When J is empty, $F_J^\pm(x)$ reduces to $F^\pm(x)$ defined in (2.2). We define $X_{\xi,J}^\pm(x)$ by

$$X_{\xi,J}^\pm(x) = \frac{\phi_\xi^\pm(xq^{\pm\rho})}{F_J^\pm(xq^{\mp(\rho+1)})F_J^\pm(xq^{\pm(\rho+1)})} \prod_{\ell=1}^{\rho} \frac{\phi_\xi^\pm(xq^{\pm(2\ell-\rho-1)})}{\phi_\xi^\pm(xq^{\pm(2\ell-\rho)})}$$
$$\times \prod_{j\in J} \left(s_j^\pm(xq^{\mp(\rho-1)})s_j^\pm(xq^{\pm(\rho-1)})\right)^{-1}. \tag{5.19}$$

Here ρ is given by $|J|$. We define $\chi_{\xi,n}^{\pm;J}$ by the following series expansion:

$$X_{\xi,J}^\pm(x) = \sum_{n=0}^{\infty} \chi_{\xi,n}^{\pm;J} x^n (|x| \ll 1). \tag{5.20}$$

We shall show in section 6 the following.

Lemma 9. *Let $t_1, t_2, \ldots, t_R$ be a set of regular Bethe roots at generic q in the inhomogeneous case. Sending centre Λ of the kN-complete string to $\pm\infty$, we have*

$$\left(\frac{1}{[N]_q!}(\hat{C}(\pm\infty))^N\right)^k B(t_1)B(t_2)\cdots B(t_R)\frac{1}{([N]_q!)^k}\hat{B}(z_1)\hat{B}(z_2)\cdots\hat{B}(z_{kN})|0\rangle$$
$$= \left(\frac{[kN]_q!}{([N]_q!)^k}\right)^2 \sum_{\rho=0}^{kN} \sum_{J\subset\Sigma_R}^{|J|=\rho} \sum_{n_0=0}^{(kN-\rho)\Theta(\rho-1)} \left(\sum_{n_1+\cdots+n_\rho=n_0} q^{\pm\sum_{j=1}^{\rho} 2j(n_j+1)} \prod_{k=1}^{\rho} \hat{b}_{\xi,n_k}^\pm\right)$$
$$\times \prod_{j\in\Sigma_R\setminus J} B(t_j)|0\rangle \exp\left(\pm\sum_{j\in J} t_j\right) q^{\mp(n_0+\rho)(\rho+1)} \left(\prod_{j\in J} a_\xi^{6V}(t_j) \prod_{\ell\in\Sigma_R\setminus J} f(t_j - t_\ell)\right)$$
$$\times (-1)^{kN}(q^{\pm 1} - q^{\mp 1})^\rho \sum_{\ell=0;\ell\leqslant\rho}^{kN-\rho-n_0} (-1)^\ell \chi_{\xi,kN-\rho-n_0-\ell}^{\pm;J} \sum_{L\subset J}^{|L|=\ell} \exp\left(\pm\sum_{j\in L} 2t_j\right)$$
$$\times \frac{1}{[\rho]_q!} \prod_{i=0}^{\ell-1} [L/2 - R - kN + i]_q \prod_{j=0}^{\rho-\ell-1} [L/2 - R - kN - j]_q + O\left(\epsilon_0^\pm\right). \tag{5.21}$$

The symbol $\sum_{n_1+\cdots+n_\rho=n_0}$ denotes the sum overall non-negative integers $n_1, n_2, \ldots, n_\rho$ such that their sum is given by n_0. Here $\Theta(x) = 1$ for $x \geqslant 0$ and $\Theta(x) = 0$ for $x < 0$.

In expansion (5.21), we call such terms with $\rho > 0$ *off-diagonal terms*, and the term with $\rho = 0$ the *diagonal term*.

Assuming conjecture 2 we have the following.

Proposition 10. *Let q_0 be a root of unity with $q_0^{2N} = 1$. In the cases of sectors A and B, we have*

$$\begin{aligned} &\left(S_\xi^{+(N)}\right)^{(k)}\left(T_\xi^{-(N)}\right)^{(k)}|R\rangle = (-1)^{kN}\tilde{\chi}^+_{\xi,kN}|R\rangle, \\ &\left(T_\xi^{+(N)}\right)^{(k)}\left(S_\xi^{-(N)}\right)^{(k)}|R\rangle = (-1)^{kN}\tilde{\chi}^-_{\xi,kN}|R\rangle, \qquad \text{for} \quad k \in \mathbf{Z}_{\geqslant 0}. \end{aligned} \tag{5.22}$$

Proof. Let us recall $S^Z = L/2 - R$. We have two cases whether $\rho \equiv 0 \pmod N$ or not. When $\rho \not\equiv 0 \pmod N$, we have the vanishing product

$$\lim_{q\to q_0}\left(\frac{1}{[\rho]_q!}\prod_{i=0}^{\ell-1}[S^Z - kN + i]_q \prod_{j=0}^{\rho-\ell-1}[S^Z - kN - j]_q\right) = 0. \tag{5.23}$$

When $\rho \equiv 0 \pmod N$ and $\rho > 0$, then $\rho = tN$ for an integer t with $0 < t \leqslant k$. We have

$$\sum_{n_1+\cdots+n_{tN}=n_0} q_0^{\pm\sum_{j=1}^{tN} 2jn_j}\prod_{k=1}^{tN}\hat{b}^{\pm}_{\xi,n_k}|0\rangle = 0, \qquad \text{for} \quad n_0 \geqslant 0. \tag{5.24}$$

We now show (5.24). When $q = q_0$, a root of unity with $q_0^{2N} = 1$, we have the vanishing product of B operators with their arguments given by a complete tN-string, as follows [33]:

$$\hat{B}(z_1)\hat{B}(z_2)\cdots\hat{B}(z_{tN})|0\rangle = 0. \tag{5.25}$$

Expanding $\hat{B}(z_1)\hat{B}(z_2)\cdots\hat{B}(z_{tN})|0\rangle$ with respect to $\epsilon_0^{\pm}$, we have

$$\sum_{n_0=0}^{\infty}\left(\epsilon_0^{\pm}\right)^{n_0}\sum_{n_1+\cdots+n_{tN}=n_0} q_0^{\pm\sum_{j=1}^{tN}2jn_j}\prod_{k=1}^{tN}\hat{b}^{\pm}_{\xi,n_k}|0\rangle = 0.$$

Therefore, all the off-diagonal terms vanish in (5.21). Sending q to q_0, a root of unity with $q_0^{2N} = 1$, we have relations (5.22) of proposition 10. □

Here we note that we can show (5.25) also by the algebraic Bethe ansatz calculating all the matrix elements of the product of B operators acting on the vacuum [24].

We obtain coefficients $\tilde{\chi}^{\pm}_{\xi,kN}$ of proposition 10 (and proposition 4), evaluating $\chi^{\pm;J}_{\xi,kN}$, which is defined by (5.20), at q_0 where we set $J = \emptyset$, $\rho = 0$ and $t_j = \tilde{t}_j$ for all j. Expression (2.4) of $\tilde{\chi}^{\pm}_{\xi,kN}$ is derived through the following series expansion with respect to small x:

$$\frac{1}{(1 - xq\exp(2t_j))(1 - xq^{-1}\exp(2t_j))} = \sum_{k=0}^{\infty}[k+1]_q x^k\,\mathrm{e}^{2kt_j}. \tag{5.26}$$

We derive expression (2.5) of eigenvalues λ_k through relations (4.35) and (4.36) for type I and II, respectively. They connect the two sets of generators, $S_\xi^{+(N)}$ and $T_\xi^{-(N)}$, and x_0^+ and x_1^-, respectively.

5.5. Derivation of theorem 3

It follows from lemma 6 that we obtain theorem 3 (and proposition 4) from propositions 8 and 10. Here we recall that by assuming conjecture 2, propositions 8 and 10 are derived from lemmas 7 and 9, which are shown in sections 6.3 and 6.4, respectively.

5.6. *On higher spin generalizations*

By introducing higher dimensional representations of L operators, the inhomogeneous transfer matrix of the six-vertex model, $\tau_{6V}(z; \{\xi_n\})$, is generalized into that acting on the tensor product of higher dimensional vector spaces.

In the cases of sectors A and B, we show that the generalized inhomogeneous transfer matrix has the sl_2 loop algebra symmetry at $q = q_0$. Here we employ the standard fusion method, and hence the higher spin generalization almost corresponds to a special case of the inhomogeneous one. The derivation of the sl_2 loop algebra symmetry for the generalized inhomogeneous transfer matrix is parallel to that of section 3. The symmetry operators are derived from the Nth powers of B and C operators by taking the infinite rapidity limit and then sending q to q_0. We then show similarly as in section 6 that all regular Bethe states at q_0 are highest weight vectors in the cases of sectors A and B.

6. Explicit derivation of regular Bethe vectors being highest weight

6.1. *Important relations for Bethe roots*

Lemma 11. *Let $t_1, t_2, \ldots, t_R$ be a set of Bethe roots at a given value of q, and S be a subset of $\Sigma_R = \{1, 2, \ldots, R\}$. For any pair of sets J_A and J_B such that $J_A \cup J_B = S$ and $J_A \cap J_B = \emptyset$, we have the following:*

$$\prod_{j\in J_A}\left(a_\xi^{6V}(t_j)\prod_{k\in\Sigma_R\backslash S} f(t_j - t_k)\right)\prod_{j\in J_A}\prod_{k\in J_B} f(t_j - t_k)$$
$$= \prod_{j\in J_A}\left(d_\xi^{6V}(t_j)\prod_{k\in\Sigma_R\backslash S} f(t_k - t_j)\right)\prod_{j\in J_A}\prod_{k\in J_B} f(t_k - t_j). \tag{6.1}$$

Furthermore, we have

$$\prod_{j\in S} a_\xi^{6V}(t_j)\prod_{k\in\Sigma_R\backslash S} f(t_j - t_k) = \prod_{j\in S} d_\xi^{6V}(t_j)\prod_{k\in\Sigma_R\backslash S} f(t_k - t_j). \tag{6.2}$$

Proof. The first relation (6.1) is derived from the Bethe ansatz equations (3.13). The second relation (6.2) follows from (6.1). □

6.2. *Fundamental formula of the algebraic BA with infinite rapidities*

Through the commutation relations such as (3.7), which are derived from the Yang–Baxter equation, it was shown in [24]:

$$C(w_0)B(w_1)\cdots B(w_n) = B(w_1)B(w_2)\cdots B(w_n)C(w_0)$$
$$+\sum_{j=1}^{n} B(w_1)\cdots B(w_{j-1})B(w_{j+1})\cdots B(w_n)g(w_0 - w_j)$$
$$\times\left\{A(w_0)D(w_j)\prod_{k\neq j} f(w_0 - w_k)f(w_k - w_j)\right.$$
$$\left. - A(w_j)D(w_0)\prod_{k\neq j} f(w_k - w_0)f(w_j - w_k)\right\}$$

$$
\begin{aligned}
&- \sum_{1\leqslant j<k\leqslant n} B(w_0)B(w_1)\cdots B(w_{j-1})B(w_{j+1})\cdots B(w_{k-1})B(w_{k+1})\cdots B(w_n)\\
&\times g(w_0-w_j)g(w_0-w_k)\{A(w_j)D(w_k)f(w_j-w_k)\\
&\times \prod_{\ell\neq j,k} f(w_j-w_\ell)f(w_\ell-w_k)\\
&+A(w_k)D(w_j)f(w_k-w_j)\prod_{\ell\neq j,k} f(w_k-w_\ell)f(w_\ell-w_j)\}. \qquad (6.3)
\end{aligned}
$$

Here parameters w_j for $j=0,1,\ldots,n$ are arbitrary.

We denote by Σ_M the set of M letters: $\Sigma_M=\{1,2,\ldots,M\}$. For a set S we express by $|S|$ the number of elements. The symbol $\mathcal{S}(n)$ denotes the symmetric group on n letters such as $\{1,2,\ldots,n\}$. For a finite set Σ we define $\mathrm{Sym}(\Sigma)$, the symmetric group acting on the set Σ, as follows. Let m be the number of elements of Σ. Then, each element of $\mathrm{Sym}(\Sigma)$ gives a one-to-one map: $\Sigma_m\to\Sigma$. For instance, we have $\mathcal{S}(n)=\mathrm{Sym}(\Sigma_n)$.

Recall that w_j for $j=1,2,\ldots,n$ are arbitrary parameters. For $S_n=\{j_1,j_2,\ldots,j_n\}\subset\Sigma_M$, we introduce the following symbols:

$$
\begin{aligned}
\alpha_\xi^{\pm;\Sigma_M\setminus S_n}(z)&=a_\xi^{6V}(z)q^{\mp L/2}\prod_{\ell\in\Sigma_M\setminus S_n} q^{\pm1}f(z-w_\ell),\\
\bar{\alpha}_\xi^{\pm;\Sigma_M\setminus S_n}(z)&=d_\xi^{6V}(z)q^{\pm L/2}\prod_{\ell\in\Sigma_M\setminus S_n} q^{\mp1}f(w_\ell-z).
\end{aligned}
\qquad (6.4)
$$

Definition 12. *Let S_n be a subset of Σ_M with n elements: $S_n=\{j_1,j_2,\ldots,j_n\}\subset\Sigma_M$. For a given $P\in\mathcal{S}(n)$, we denote by S_ℓ^P the set $\{j_{P1},\ldots,j_{P\ell}\}$ for $\ell=1,2,\ldots,n$. Then we define $\Delta(\xi)^\pm_{S_n;\Sigma_M}$ by*

$$
\begin{aligned}
\Delta(\xi)^\pm_{S_n;\Sigma_M}=\sum_{P\in\mathcal{S}(n)}\prod_{\ell=1}^{n}\Bigg(&\alpha_\xi^{\pm;\Sigma_M\setminus S_n}(w_{j_{P\ell}})\prod_{k\in S_n\setminus S_\ell^P} q^{\pm1}f\left(w_{j_{P\ell}}-w_k\right)\\
&-\bar{\alpha}_\xi^{\pm;\Sigma_M\setminus S_n}(w_{j_{P\ell}})\prod_{k\in S_n\setminus S_\ell^P} q^{\mp1}f\left(w_k-w_{j_{P\ell}}\right)\Bigg). \qquad (6.5)
\end{aligned}
$$

Here we note that $\Sigma_M\Big\backslash S_\ell^P=(\Sigma_M\backslash S_n)\cup\left(S_n\Big\backslash S_\ell^P\right)$, for $P\in\mathcal{S}(n)$ and $\ell=1,2,\ldots,n$. We shall sometimes express $\Delta(\xi)^\pm_{S_n;\Sigma_M}$ as $\Delta(\xi)^{\pm;\Sigma_M\setminus S_n}_{S_n}$.

Lemma 13. *Let w_j be arbitrary parameters for $j\in\Sigma_M$. We have the following:*

$$
(\hat{C}(\pm\infty))^n\prod_{\ell\in\Sigma_M}B(w_\ell)|0\rangle=\sum_{S_n\subset\Sigma_M}^{|S_n|=n}\prod_{\ell\in\Sigma_M\setminus S_n}B(w_\ell)|0\rangle\exp\left(\pm\sum_{j\in S_n}w_j\right)\Delta(\xi)^\pm_{S_n;\Sigma_M}. \qquad (6.6)
$$

Here $\sum_{S_n\subset\Sigma_M}^{|S_n|=n}$ denotes the sum over all such subsets S_n of Σ_M that have n elements.

Proof. Formula (6.6) is derived through induction on n. First, sending w_0 to infinity in equation (6.3), we have the following:

$$
\begin{aligned}
\hat{C}(\pm\infty)B(w_1)\cdots B(w_M)|0\rangle&=\sum_{j=1}^{M}B(w_1)\cdots B(w_{j-1})B(w_{j+1})\cdots B(w_M)|0\rangle\,\mathrm{e}^{\pm w_j}\\
&\times\left(a_\xi^{6V}(w_j)q^{\mp L/2}\prod_{k\neq j}q^{\pm1}f(w_j-w_k)-d_\xi^{6V}(w_j)q^{\pm L/2}\prod_{k\neq j}q^{\mp1}f(w_k-w_j)\right). \qquad (6.7)
\end{aligned}
$$

This gives the case of $n = 1$. Let us assume the case of n. Multiplying both hand sides of equation (6.6) in the case of n by $\hat{C}(\pm\infty)$, applying equation (6.7) to the product of B operators on the right-hand side, we have equation (6.6) in the case of $n + 1$. □

Let m and n be non-negative integers satisfying $m \geqslant n$. We define the q-binomial coefficient by

$$\begin{bmatrix} m \\ n \end{bmatrix}_q = \frac{[m]_q}{[m-n]_q [n]_q!}. \tag{6.8}$$

Lemma 14. *Let S_n be a subset of $\Sigma_M = \{1, 2, \ldots, M\}$ with n integers. We express it as $S_n = \{j_1, j_2, \ldots, j_n\}$. Let w_j for $j = 1, 2, \ldots, n$ be arbitrary parameters. We have*

$$\Delta(\xi)^{\pm}_{S_n;\Sigma_M} = \sum_{P\in\mathcal{S}(n)} \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\pm n(n-1)/2} q^{\mp(n-1)k} \prod_{1\leqslant \ell\leqslant n-k} \alpha_\xi^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{P\ell}}\right)$$
$$\times \prod_{n-k<\ell\leqslant n} \bar{\alpha}_\xi^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{P\ell}}\right) \prod_{1\leqslant \ell<m\leqslant n} f\left(w_{j_{P\ell}} - w_{j_{Pm}}\right). \tag{6.9}$$

The proof of lemma 14 will be given in appendix C.

6.3. Proof of lemma 7

Proof. In formula (6.9) we put $n = N_c$ and set parameters w_j as

$$w_j = t_j \qquad \text{for} \quad j = 1, 2, \ldots, R. \tag{6.10}$$

Let us specify permutation $P \in \mathcal{S}(N_c)$ as follows. First, we define disjoint sets I and K by $I = \{P1, P2, \ldots, P(N_c - k)\}$ and $K = \{P(N_c - k + 1), \ldots, P(N_c - 1), PN_c\}$. Here we have $I \cup K = \Sigma_{N_c} = \{1, 2, \ldots, N_c\}$. Second, we define $P_I \in \mathrm{Sym}(I)$ by $P_I j = Pj$ for $j = 1, 2, \ldots, N_c - k$, and $P_K \in \mathrm{Sym}(K)$ by $P_K j = P(j + (N_c - k))$ for $j = 1, 2, \ldots, k$. Then, permutation P is given by $Pj = P_I j$ for $1 \leqslant j \leqslant N_c - k$ and $Pj = P_K(j - (N_c - k))$ for $N_c - k + 1 \leqslant j \leqslant N_c$. We therefore express the sum over all permutations as follows:

$$\sum_{P\in\mathcal{S}(N_c)} = \sum_{I\cup K=\Sigma_{N_c}}^{|I|=N_c-k,|K|=k} \sum_{P_I\in\mathrm{Sym}(I)} \sum_{P_K\in\mathrm{Sym}(K)}. \tag{6.11}$$

Here, $\sum_{I\cup K=\Sigma_{N_c}}^{|I|=N_c-k,|K|=k}$ denotes the sum over all decompositions of Σ_{N_c} into disjoint sets I and K, where the numbers of elements are fixed such that $|I| = N_c - k$ and $|K| = k$. Let J_I and J_K be $J_I = \{j_\ell | \ell \in I\}$ and $J_K = \{j_\ell | \ell \in K\}$, respectively. We have

$$\prod_{\ell=1}^{N_c-k} \alpha_\xi^{\pm;\Sigma_R\setminus S_{N_c}}\left(w_{j_{P_I\ell}}\right) = \prod_{j\in J_I} \alpha_\xi^{\pm;\Sigma_R\setminus S_{N_c}}(t_j) \qquad \text{for} \quad P_I \in \mathrm{Sym}(I)$$

Similarly, we have

$$\prod_{\ell=N_c-k+1}^{N_c} \bar{\alpha}_\xi^{\pm;\Sigma_R\setminus S_{N_c}}\left(w_{j_{P_K(\ell-(N_c-k))}}\right) = \prod_{j\in J_K} \bar{\alpha}_\xi^{\pm;\Sigma_R\setminus S_{N_c}}(t_j) \qquad \text{for} \quad P_K \in \mathrm{Sym}(K)$$

Thus, $\Delta(\xi)^{\pm}_{S_{N_c};\Sigma_R} q^{\mp N_c(N_c-1)/2}$ is given by

$$\sum_{k=0}^{N_c} (-1)^k \begin{bmatrix} N_c \\ k \end{bmatrix}_q q^{\mp(N_c-1)k} \sum_{I\cup K=\Sigma_{N_c}}^{|I|=N_c-k,|K|=k} \prod_{j\in J_I} \alpha_\xi^{\pm;\Sigma_R\backslash S_{N_c}}(t_j)$$
$$\times \prod_{\ell\in J_K} \bar{\alpha}_\xi^{\pm;\Sigma_R\backslash S_{N_c}}(t_\ell) \prod_{j\in J_I}\prod_{\ell\in J_K} f(t_j - t_\ell)$$
$$\times \sum_{P_I\in \mathrm{Sym}(I)} \prod_{1\leqslant \ell<m\leqslant N-k} f\left(t_{j_{P_I\ell}} - t_{j_{P_I m}}\right) \sum_{P_K\in \mathrm{Sym}(K)} \prod_{1\leqslant \ell<m\leqslant k} f\left(t_{j_{P_K\ell}} - t_{j_{P_K m}}\right).$$

Applying formula (B.2) to the sums over $\mathrm{Sym}(I)$ and $\mathrm{Sym}(K)$, we have factors $[N_c-k]_q!$ and $[k]_q!$, respectively. Thus, we have

$$\Delta(\xi)^{\pm}_{S_{N_c};\Sigma_R} = q^{\pm N_c(N_c-1)/2}[N_c]_q! \sum_{k=0}^{N_c} (-1)^k q^{\mp(N_c-1)k}$$
$$\times \sum_{I\cup K=\Sigma_{N_c}}^{|I|=N_c-k,|K|=k} \prod_{j\in J_I} \alpha_\xi^{\pm;\Sigma_R\backslash S_{N_c}}(t_j) \prod_{\ell\in J_K} \bar{\alpha}_\xi^{\pm;\Sigma_R\backslash S_{N_c}}(t_\ell) \left(\prod_{j\in J_I}\prod_{\ell\in J_K} f(t_j-t_\ell)\right).$$

We apply lemma 11 to the product of α_ξ's or $\bar{\alpha}_\xi$'s, and we make use of formula (B.3) where S_{N_c} and J_K give Σ_m and S_n of (B.3), respectively. Through the q-binomial formula (B.1), we obtain (5.11) for generic q. □

6.4. Derivation of lemma 9

6.4.1. A complete kN-string as additional rapidities. Let us recall that the action of $\left(S_\xi^{+(N)}\right)^k \left(T_\xi^{-(N)}\right)^k$ on the Bethe state $|R\rangle$ is formulated through (5.14), where $t_1, t_2, \ldots, t_R$ are regular Bethe roots at generic q, and $z_1, z_2, \ldots, z_{kN}$ are additional kN rapidities forming a complete kN-string: $z_j = \Lambda + (kN + 1 - 2j)\eta$ for $j = 1, 2, \ldots, kN$.

We now consider a complete N_c-string, where N_c corresponds to the number of rapidities in the complete string, i.e. $N_c = kN$. Hereafter, we set parameters w_j as follows:

$$w_j = t_j \qquad \text{for} \quad 1 \leqslant j \leqslant R; \qquad w_{j+R} = z_j \qquad \text{for} \quad 1 \leqslant j \leqslant N_c. \tag{6.12}$$

We write index $R + j$ as $\underline{j}$, i.e. $w_{\underline{j}} = w_{R+j}$ for $1 \leqslant j \leqslant N_c$. Here $\Sigma_R = \{1, 2, \ldots, R\}$ gives the set of indices of Bethe roots, $t_1, t_2, \ldots, t_R$. We denote by Z_{N_c} the set of indices of N_c rapidities z_j:

$$Z_{N_c} = \{R + 1, R + 2, \ldots, R + N_c\} = \{\underline{1}, \underline{2}, \ldots, \underline{N_c}\}. \tag{6.13}$$

The set of all indices, $\Sigma_{R+N_c} = \{1, 2, \ldots, R, R + 1, \ldots, R + N_c\}$, is given by the union of Σ_R and Z_{N_c}, i.e. $\Sigma_{R+N_c} = \Sigma_R \cup Z_{N_c}$.

Let S_{N_c} be a subset of Σ_{R+N_c} with N_c elements. We define two disjoint sets J and W by $J = S_{N_c} \cap \Sigma_R$ and $W = S_{N_c} \cap Z_{N_c}$, respectively. Let ρ be the number of elements of J, i.e. $\rho = |J|$. We express elements of J as j_ℓ for $\ell = 1, 2, \ldots, \rho$, and put them in increasing order: $j_1 < j_2 < \cdots < j_\rho$. Set W is given by $S_{N_c}\backslash J$, and it is the set of suffices for rapidities z_j in S_{N_c}. In order to specify subset $S_{N_c} = J \cup W$, we specify set $Z_{N_c}\backslash W$. We express elements of $Z_{N_c}\backslash W$ by $\underline{\nu_1}, \underline{\nu_2}, \ldots, \nu_\rho$, and put them in increasing order: $\nu_1 < \nu_2 < \cdots < \nu_\rho$. Here we note that $\Sigma_{R+N_c}\backslash S_{N_c} = (\Sigma_R\backslash J) \cup (Z_{N_c}\backslash W)$.

We normalize $\Delta(\xi)^{\pm}_{S_{N_c};\Sigma_{R+N_c}}$ as $\hat{\Delta}(\xi)^{\pm}_{S_{N_c};\Sigma_{R+N_c}} = \Delta(\xi)^{\pm}_{S_{N_c};\Sigma_{R+N_c}} \big/ \prod_{\underline{\nu}\in W} n_\xi(z_\nu)$. We express $\hat{\Delta}(\xi)^{\pm}_{S_{N_c};\Sigma_{R+N_c}}$ as $\hat{\Delta}(\xi)^{\pm;\nu_1,\ldots,\nu_\rho}_{j_1,\ldots,j_\rho}$, for simplicity. We have

$$\hat{\Delta}(\xi)^{\pm;\Sigma_{R+N_c}\backslash S_{N_c}}_{S_{N_c}} = \hat{\Delta}(\xi)^{\pm;(\Sigma_R\backslash J)\cup(Z_{N_c}\backslash W)}_{S_{N_c}} = \hat{\Delta}(\xi)^{\pm;\nu_1,\nu_2,\ldots,\nu_\rho}_{j_1,j_2,\ldots,j_\rho}. \tag{6.14}$$

Hereafter we write ϵ_0^+ simply as ϵ_0 and $\hat{\Delta}(\xi)^{+;\nu_1,\nu_2,\ldots,\nu_\rho}_{j_1,j_2,\ldots,j_\rho}$ as $\hat{\Delta}(\xi)^{\nu_1,\nu_2,\ldots,\nu_\rho}_{j_1,j_2,\ldots,j_\rho}$.

Taking advantage of the complete N_c-string z_j's, we can show the following.

Lemma 15. *If q is generic and $\rho > 0$, $\hat{\Delta}(\xi)^{\nu_1,\ldots,\nu_\rho}_{j_1,\ldots,j_\rho}$ vanishes unless $\nu_1, \nu_2, \ldots, \nu_\rho$ are given by $\nu+1, \nu+2, \ldots, \nu+\rho$, respectively, for an integer ν with $0 \leqslant \nu \leqslant N_c - \rho$.*

Lemma 15 will be shown in appendix D.

Substituting n and M of formula (6.6) by N_c and $R+N_c$, respectively, we have for generic q the following:

$$\begin{aligned}
(\hat{C}(\infty))^{N_c} B(t_1)\cdots B(t_R)\hat{B}(z_1)\cdots\hat{B}\left(z_{N_c}\right)|0\rangle &= \sum_{\rho=0}^{N_c}\sum_{J\subset\Sigma_R}^{|J|=\rho}\sum_{W\subset Z_{N_c}}^{|W|=N_c-\rho} \\
&\times \prod_{\underline{\nu}\in Z_{N_c}\backslash W}\hat{B}(z_\nu)\prod_{\ell\in\Sigma_R\backslash J}B(t_\ell)|0\rangle \exp\left(\sum_{j\in J}t_j\right)\frac{\exp\left(\sum_{\underline{\nu}\in W}2z_\nu\right)}{(q-q^{-1})^{N_c-\rho}}\hat{\Delta}(\xi)^{+;\Sigma_{R+N_c}\backslash S_{N_c}}_{S_{N_c}} \\
&= \sum_{\rho=0}^{N_c}(\epsilon_0)^{\rho-N_c}\sum_{1\leqslant j_1<\cdots<j_\rho\leqslant R}\ \sum_{1\leqslant\nu_1<\cdots<\nu_\rho\leqslant N_c}\hat{B}\left(z_{\nu_1}\right)\cdots\hat{B}\left(z_{\nu_\rho}\right)\prod_{\ell\in\Sigma_R\backslash J}B(t_\ell)|0\rangle \\
&\times q^{2\nu_1+\cdots+2\nu_\rho}\hat{\Delta}(\xi)^{\nu_1,\ldots,\nu_\rho}_{j_1,\ldots,j_\rho}\times \mathrm{e}^{\sum_{j\in J}t_j}\frac{q^{-N_c(N_c+1)}}{(q-q^{-1})^{N_c-\rho}}.
\end{aligned}\tag{6.15}$$

Here we assume that when $\rho = 0$, the sums $\sum_{1\leqslant\nu_1<\cdots<\nu_\rho\leqslant N_c}$ and $\sum_{J\subset\Sigma_R}$ are given by 1, respectively. Recall that the term of $\rho = 0$ in (6.15) is called the diagonal term, and the terms of $\rho > 0$ are called off-diagonal terms.

Proposition 16. *Let $t_1, t_2, \ldots, t_R$ be regular Bethe roots at generic q, and $z_1, z_2, \ldots, z_{N_c}$ form a complete N_c-string with centre Λ. We have*

$$\begin{aligned}
(\hat{C}(\infty))^{N_c} B(t_1)\cdots B(t_R)\hat{B}(z_1)\cdots\hat{B}\left(z_{N_c}\right)|0\rangle &= \prod_{\ell=1}^{R}B(t_\ell)|0\rangle\epsilon_0^{-N_c}\hat{\Delta}(\xi)^{+;\Sigma_{R+N_c}\backslash Z_{N_c}}_{Z_{N_c}}\frac{q^{-N_c(N_c+1)}}{(q-q^{-1})^{N_c}} \\
&+\sum_{\rho=1}^{N_c}\sum_{J\subset\Sigma_R}^{|J|=\rho}\sum_{n_0=0}^{\infty}\epsilon_0^{n_0+\rho-N_c}\sum_{n_1+\cdots+n_\rho=n_0}\hat{b}^+_{\xi,n_1}\cdots\hat{b}^+_{\xi,n_\rho}\prod_{j\in\Sigma_R\backslash J}B(t_j)|0\rangle \\
&\times\Sigma(\hat{\Delta}_\xi)_J\times\mathrm{e}^{\sum_{j\in J}t_j}\frac{q^{-N_c(N_c+1)}}{(q-q^{-1})^{N_c-\rho}}q^{\sum_{\ell=1}^{\rho}2\ell(n_\ell+1)},
\end{aligned}\tag{6.16}$$

where $\Sigma(\hat{\Delta}_\xi)_J$ is given by

$$\Sigma(\hat{\Delta}_\xi)_J = \sum_{\nu=0}^{N_c-\rho}q^{2\nu(n_0+\rho)}\hat{\Delta}(\xi)^{\nu+1,\ldots,\nu+\rho}_{j_1,\ldots,j_\rho}. \tag{6.17}$$

Here, $\sum_{J\subset\Sigma_R}^{|J|=\rho}$ denotes the sum over all such subsets of Σ_R that have ρ elements, where $J = \{j_1,\ldots,j_\rho\}$, and $\sum_{n_1+\cdots+n_\rho=n_0}$ the sum over all non-negative integers $n_1, n_2, \ldots, n_\rho$ satisfying the condition: $n_1+n_2+\cdots+n_\rho = n_0$.

Proof. For off-diagonal terms in (6.15), we expand the product of B operators, $\hat{B}\left(z_{\nu_1}\right)\cdots\hat{B}\left(z_{\nu_\rho}\right)$, in the power series of ϵ_0 through (5.16). We have the infinite sum over all non-negative integers $n_1, n_2, \ldots, n_\rho$ satisfying the condition: $n_1 + n_2 + \cdots + n_\rho = n_0$. It follows from lemma 15 that we have

$$\sum_{1\leqslant \nu_1 < \cdots < \nu_\rho \leqslant N_c} q^{2\nu_1(n_1+1)+\cdots+2\nu_\rho(n_\rho+1)} \hat{\Delta}(\xi)_{j_1,\ldots,j_\rho}^{\nu_1,\ldots,\nu_\rho} = q^{\sum_{\ell=1}^{\rho} 2\ell(n_\ell+1)} \Sigma(\Delta_\xi)_J.$$

Making use of it, we derive expression (6.16). □

Let us consider the term of $\rho = 0$ in expansion (6.15). It leads to the eigenvalue of $\left(S_\xi^{+(N)}\right)^k \left(T_\xi^{-(N)}\right)^k$ by putting $N_c = kN$ and sending q to a root of unity q_0.

Lemma 17. *The coefficient of the diagonal term of (6.15) is given by*

$$\epsilon_0^{-N_c} \frac{q^{-N_c(N_c+1)}}{(q-q^{-1})^{N_c}} \hat{\Delta}(\xi)_{Z_{N_c}}^{+;\Sigma_{R+N_c}\setminus Z_{N_c}} = (-1)^{N_c} \chi_{\xi,N_c}^{+} ([N_c]_q!)^2 + O(\epsilon_0). \tag{6.18}$$

Proof. Making use of formula (6.9) at generic q, we have

$$\begin{aligned}\hat{\Delta}(\xi)_{Z_{N_c}}^{+;\Sigma_{R+N_c}\setminus Z_{N_c}} &= [N_c]_q! \frac{\prod_{\ell=1}^{N_c-1} \phi_\xi^+(\epsilon_0 q^{2\ell+1})}{\prod_{\ell=1}^{N_c} \phi_\xi^+(\epsilon_0 q^{2\ell})} F^+(\epsilon_0) F^+(\epsilon_0 q^{2N_c+2}) \\ &\quad \times \sum_{j=0}^{N_c} (-1)^j \begin{bmatrix} N_c \\ j \end{bmatrix}_q q^{N_c(N_c-1)/2-(N_c-1)j} \frac{\phi_\xi^+(\epsilon_0 q^{2j+1})}{F^+(\epsilon_0 q^{2j}) F^+(\epsilon_0 q^{2j+2})}. \end{aligned} \tag{6.19}$$

We expand the last line of (6.19) in terms of ϵ_0. It is given by the series of (5.20) with $J = \emptyset$ and $\rho = 0$. By noting $\prod_{\ell=0}^{N_c-1}(1 - q^{2m-2\ell}) = 0$ for $0 \leqslant m < N_c$, $\hat{\Delta}(\xi)_{Z_{N_c}}^{+;\Sigma_{R+N_c}\setminus Z_{N_c}}$ is given by

$$\epsilon_0^{N_c} \chi_{\xi,N_c}^{+} (-1)^{N_c} q^{N_c(N_c+1)} ([N_c]_q!)^2 (q-q^{-1})^{N_c} + O\left(\epsilon_0^{N_c+1}\right). \tag{6.20}$$

□

We now consider the off-diagonal terms of (6.15). Let J_I and J_K be disjoint sets such that $J_I \cup J_K = J$, where $|J| = \rho$. We define $G_{J_I,J_K}^{\pm}(x)$ by

$$G_{J_I,J_K}^{\pm}(x) = \prod_{j\in J_I} s_j^{\pm}(x q^{\mp(\rho-1)}) \prod_{j\in J_K} s_j^{\pm}(x q^{\pm(\rho-1)}). \tag{6.21}$$

Let us define subsequences. We consider two sequences of numbers, $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$, and assume that $\{b_1, b_2, \ldots, b_n\} \subset \{a_1, a_2, \ldots, a_m\}$ and $n \leqslant m$. We define sequence $i(j)$ by $a_{i(j)} = b_j$ for $j = 1, 2, \ldots, n$. We say that $b_1, b_2, \ldots, b_n$ is a subsequence of $a_1, a_2, \ldots, a_m$, if $1 \leqslant i(1) < i(2) < \cdots < i(n) \leqslant m$.

Proposition 18. *Let J be a subset of Σ_R. We denote it as follows: $J = \{j_1, j_2, \ldots, j_\rho\}$, where $j_1 < j_2 < \cdots < j_\rho$. Then, $\Sigma(\hat{\Delta}_\xi)_J$ is evaluated at generic q as follows:*

$$\begin{aligned}\sum_{\nu=0}^{N_c-\rho} q^{2\nu(n_0+\rho)} \hat{\Delta}(\xi)_{j_1,\ldots,j_\rho}^{\nu+1,\ldots,\nu+\rho} &= [N_c]_q! \begin{bmatrix} N_c \\ \rho \end{bmatrix}_q q^{N_c(N_c-1)/2} q^{-(S^Z-N_c)\rho} \\ &\quad \times \prod_{j\in J}\left(a_\xi^{6\mathrm{V}}(t_j) \prod_{\ell\in\Sigma_R\setminus J} f(t_j - t_\ell)\right) F_J^+\left(\epsilon_{N_c+1}\right) F_J^+(\epsilon_0) \frac{\prod_{\ell=1}^{N_c-1}\phi_\xi^+(\epsilon_\ell q)}{\prod_{\ell=1}^{N_c}\phi_\xi^+(\epsilon_\ell)} \\ &\quad \times \sum_{\sigma=0}^{\rho} (-1)^\sigma q^{2(S^Z-N_c)\sigma} q^{-(\rho-1)\sigma} \sum_{J_I\cup J_K=J}^{|J_I|=\rho-\sigma, |J_K|=\sigma} \left(\prod_{j\in J_I}\prod_{\ell\in J_K} f(t_\ell - t_j)\right)\end{aligned}$$

$$\times \prod_{j\in J_I} s_j^+(\epsilon_0) \prod_{j\in J_K} s_j^+\left(\epsilon_{N_c+1}\right)$$

$$\times \sum_{\nu=0}^{N_c-\rho} (-1)^\nu q^{-(N_c-\rho-1-2n_0)\nu} \begin{bmatrix} N_c-\rho \\ \nu \end{bmatrix}_q X^+_{\xi,J}(\epsilon_\nu q^{\rho+1}) G^+_{J_I,J_K}(\epsilon_\nu q^{\rho+1}). \tag{6.22}$$

Here, $\sum_{J_I\cup J_K=J}^{|J_I|=\rho-\sigma,|J_K|=\sigma}$ denotes the sum over all pairs of disjoint sets J_I and J_K such that the sum of J_I and J_K gives J, and J_I and J_K have $\rho-\sigma$ and σ elements, respectively.

Proof. When $(\nu_1, \nu_2, \ldots, \nu_\rho) = (\nu+1, \nu+2, \ldots, \nu+\rho)$, we have $Z_{N_c}\backslash W = \{\underline{\nu+1}, \underline{\nu+2}, \ldots, \underline{\nu+\rho}\}$. The set $S_{N_c} = J\cup W$ is given by the following:

$$S_{N_c} = \{j_1, j_2, \ldots, j_\rho\} \cup \{\underline{1}, \underline{2}, \ldots, \underline{\nu}, \underline{\nu+\rho+1}, \ldots, \underline{N_c-1}, \underline{N_c}\}. \tag{6.23}$$

Let us put elements of S_{N_c} in increasing order as $i_1 < i_2 < \cdots < i_{N_c}$. The first ρ elements $i_1, i_2, \ldots, i_\rho$ are given by $j_1, j_2, \ldots, j_\rho$, respectively, and $i_{\rho+1}, i_{\rho+2}, \ldots, i_{\rho+\nu}$ by $\underline{1}, \underline{2}, \ldots, \underline{\nu}$, respectively, and $i_{\rho+\nu+1}, i_{\rho+\nu+2}, \ldots, i_{N_c}$ by $\underline{\rho+\nu+1}, \ldots, \underline{N_c-1}, \underline{N_c}$, respectively. By making use of formula (6.9) with S_{N_c} given by (6.23), $\hat{\Delta}(\xi)^{\nu+1,\ldots,\nu+\rho}_{j_1,\ldots,j_\rho}$ is expressed as follows:

$$\sum_{\kappa=\nu}^{\nu+\rho} (-1)^\kappa q^{N_c(N_c-1)/2-(N_c-1)\kappa} \begin{bmatrix} N_c \\ \kappa \end{bmatrix}_q \left(\prod_{\beta=1}^{\nu} n_\xi(z_\beta) \prod_{\gamma=\nu+\rho+1}^{N_c} n_\xi(z_\gamma) \right)^{-1}$$
$$\times \sum_{P\in\mathcal{S}(N_c)} \prod_{\ell=1}^{N_c-\kappa} \alpha_\xi^{+;\Sigma_{R+N_c}\backslash S_{N_c}}\left(w_{i_{P\ell}}\right) \prod_{\ell=N_c-\kappa+1}^{N_c} \bar{\alpha}_\xi^{+;\Sigma_{R+N_c}\backslash S_{N_c}}\left(w_{i_{P\ell}}\right)$$
$$\times \prod_{1\leqslant\ell<m\leqslant N_c} f\left(w_{i_{P\ell}} - w_{i_{Pm}}\right). \tag{6.24}$$

In (6.24), the range of κ has been reduced from $\kappa = 0, 1, \ldots, N_c$ into $\kappa = \nu, \nu+1, \ldots, \nu+\rho$. We first note that $f(z_j - z_{j+1}) = 0$ for $j = 1, \ldots, \nu-1$ and for $j = \nu+\rho, \ldots, N_c-1$. Hence the product $\prod_{\ell<m} f_{i_{P\ell},i_{Pm}}$ vanishes unless sequence $i_{P1}, i_{P2}, \ldots, i_{PN}$ contains two decreasing subsequences $\underline{N_c}, \underline{N_c-1}, \ldots, \underline{\nu+\rho+1}$ and $\underline{\nu}, \underline{\nu-1}, \ldots, \underline{1}$, i.e. unless $P^{-1}\ell > P^{-1}m$ for $\rho+\nu+1 \leqslant \ell < m \leqslant N_c$ and $P^{-1}\ell > P^{-1}m$ for $\rho+1 \leqslant \ell < m \leqslant \rho+\nu$. Here we define I and K by $I = \{P1, P2, \ldots, P(N_c-\kappa)\}$ and $K = \{P(N_c-\kappa+1), \ldots, P(N_c-1), PN_c\}$, respectively. Secondly, we show that the summand of equation (6.24) vanishes unless $\kappa \geqslant \nu$. We note that $\alpha_\xi^{+;\Sigma_{R+N_c}\backslash S_{N_c}}(z_\nu) = 0$ for S_{N_c} of (6.23), and $\underline{\nu} = i_{\rho+\nu}$. The summand of (6.24) therefore vanishes if $\rho+\nu \in I$, and we consider only such P where subsequence $\rho+\nu, \rho+\nu-1, \ldots, \rho+1$ is contained in sequence $P(N_c-\kappa+1), \ldots, P(N_c-1), PN_c$. We therefore have $\kappa \geqslant \nu$. Thirdly, we show that the summand of (6.24) vanishes unless $\kappa \leqslant \nu+\rho$, in the same way as the case of $\kappa \geqslant \nu$. Here we recall that $\bar{\alpha}_\xi^{+;\Sigma_{R+N_c}\backslash S_{N_c}}(z_{\nu+\rho+1}) = 0$ for S_{N_c} of (6.23), and $i_{\rho+\nu+1} = \underline{\rho+\nu+1}$. Thus, we have shown the reduction of the range of κ. Furthermore, we have shown that the summand of (6.24) vanishes unless

$$\{\rho+\nu+1, \ldots, N_c-1, N_c\} \subset I, \qquad \{\rho+1, \rho+2, \ldots, \rho+\nu\} \subset K. \tag{6.25}$$

We now consider the sum over $P \in \mathcal{S}(N_c)$ in (6.24). Let us introduce disjoint subsets S_I and S_K of S_{N_c} as follows: $S_I = \left\{i_{P1}, i_{P2}, \ldots, i_{P(N_c-\kappa)}\right\}$ and $S_K = \left\{i_{P(N_c-\kappa+1)}, \ldots, i_{P(N_c-1)}, i_{PN_c}\right\}$. Then, we have

$$S_I = J_I \cup \{\underline{N_c}, \underline{N_c-1}, \ldots, \underline{\nu+\rho+1}\}, \qquad S_K = J_K \cup \{\underline{\nu}, \ldots, \underline{2}, \underline{1}\}. \tag{6.26}$$

Here, J_I and J_K are disjoint subsets of J such that $J = J_I \cup J_K$. Let us denote by σ the number of elements of J_K, i.e. $|J_K| = \sigma$ and $|J_I| = \rho-\sigma$. We express $\sum_{P\in\mathcal{S}(N_c)}$ of (6.24) by the sum over disjoint sets I and K with $I \cup K = \Sigma_{N_c}$, such as shown in (6.11). It follows from (6.26) that the sum $\sum_{I\cup K}^{|I|=N_c-\kappa,|K|=\kappa}$ reduces to the sum $\sum_{J_I\cup J_K=J}^{|J_I|=\rho-\sigma,|J_K|=\sigma}$. Here we note that $\kappa = \sigma + \nu$. Similarly, as in the proof of lemma 7, calculating the sums over $P_I \in \mathrm{Sym}(I)$ and over $P_K \in \mathrm{Sym}(K)$, we have factors $[N_c - \kappa]!$ and $[\kappa]!$, respectively. Applying formula (6.1) of lemma 11, we show the following:

$$\prod_{j\in J_I} \alpha_\xi^{+;\Sigma_R\setminus J}(t_j) \prod_{k\in J_K} \bar{\alpha}_\xi^{+;\Sigma_R\setminus J}(t_k) \prod_{j\in J_I}\prod_{k\in J_K} f(t_j - t_k)$$
$$= q^{(L/2-R+\rho)(2\sigma-\rho)} \left(\prod_{j\in J} a_\xi^{6\mathrm{V}}(t_j) \prod_{\ell\in\Sigma_R\setminus J} f(t_j - t_\ell) \right) \prod_{j\in J_I}\prod_{k\in J_K} f(t_j - t_k).$$

Expressing sum (6.24) over κ as that of σ $(\kappa = \sigma + \nu)$, we obtain expression (6.22). □

Let us expand $G^{\pm}_{J_I,J_K}(\epsilon)$ with respect to small parameter ϵ:

$$G^{\pm}_{J_I,J_K}(\epsilon) = \sum_{\ell=0}^{\rho} (-1)^\ell G_\ell^{\pm;J_I,J_K} \epsilon^\ell. \qquad (6.27)$$

The coefficients $G_\ell^{\pm;J_I,J_K}$ for $\ell \leqslant \rho$ are explicitly given by

$$G_\ell^{\pm;J_I,J_K} = \sum_{\ell_I=0}^{\ell} q^{\pm(\rho-1)(\ell_K-\ell_I)} \sum_{L_I\subset J_I}^{|L_I|=\ell_I} \exp\left(\pm\sum_{j\in L_I} 2t_j\right) \sum_{L_K\subset J_K}^{|L_K|=\ell_K} \exp\left(\pm\sum_{j\in L_K} 2t_j\right). \qquad (6.28)$$

Here $\ell_K = \ell - \ell_I$, and $G_\ell^{\pm;J_I,J_K} = 0$ for $\ell > \rho$.

Lemma 19. *Sum (6.17),* $\Sigma(\hat{\Delta}_\xi)_J = \sum_{\nu=0}^{N-\rho} q^{2\nu(n_0+\rho)} \hat{\Delta}_{j_1,\ldots,j_\rho}^{\nu+1,\ldots,\nu+\rho}$*, is expanded in terms of* ϵ_0 *as follows:*

$$(\epsilon_0)^{N_c-\rho-n_0} (-1)^{N_c} q^{N_c(N_c+1)-(\rho+1)(n_0+\rho)} (q-q^{-1})^{N_c} ([N_c]_q!)^2$$
$$\times \left(\prod_{j\in J} a_\xi^{6\mathrm{V}}(t_j) \prod_{\ell\in\Sigma_R\setminus J} f(t_j - t_\ell) \right) \sum_{\ell=0,\ell\leqslant N_c-\rho-n_0}^{\rho} (-1)^\ell \chi^{+;J}_{\xi,N_c-\rho-n_0-\ell}$$
$$\times \frac{1}{[\rho]_q!} \prod_{i=0}^{\ell-1} [S^Z - N_c + i]_q \prod_{j=0}^{\rho-\ell-1} [S^Z - N_c - j]_q \sum_{L\subset J}^{|L|=\ell} \exp\left(\sum_{j\in L} 2t_j\right)$$
$$+ O((\epsilon_0)^{N_c-\rho-n_0+1}). \qquad (6.29)$$

We note that all the possibly divergent terms with order of $\epsilon_0^{n_0+\rho-N_c}$ in (6.16) do not diverge since $\Sigma(\hat{\Delta}_\xi)_J$ is of order of $(\epsilon_0)^{N_c-\rho-n_0}$ in the limit $\epsilon_0 \to 0$.

Proof. We evaluate sum (6.17) over ν, $\Sigma(\hat{\Delta}_\xi)_J$, by (B.1) as follows:

$$\sum_{\nu=0}^{N_c-\rho} (-1)^\nu q^{-(N_c-\rho-1-2n_0)\nu} \begin{bmatrix} N_c-\rho \\ \nu \end{bmatrix}_q X^+_{\xi,J}(\epsilon_\nu q^{\rho+1}) G^+_{J_I,J_K}(\epsilon_\nu q^{\rho+1})$$
$$= (\epsilon_0)^{N_c-\rho-n_0} (-1)^{N_c-\rho} (q-q^{-1})^{N_c-\rho} q^{-n_0(\rho+1)+(N_c-\rho)(N_c+\rho+3)/2}$$
$$\times [N_c-\rho]_q! \sum_{\ell=0}^{\rho} (-1)^\ell \chi^{+;J}_{\xi,N_c-\rho-n_0-\ell} G_\ell^{+;J_I,J_K} + O((\epsilon_0)^{N_c-\rho-n_0+1}). \qquad (6.30)$$

Here, the product $\prod_{i=0}^{N_c-\rho-1}(1-q^{2(j+\ell+n_0-i)})$ vanishes for $j+\ell < N_c-\rho-n_0$, and is given by $(-1)^{N_c-\rho}(q-q^{-1})^{N_c-\rho}[N_c-\rho]_q! \times q^{(N_c-\rho)(N_c-\rho+1)/2}$ for $j+\ell = N_c-\rho-n_0$. Sum (6.17), $\Sigma(\hat{\Delta}_\xi)_J$, is thus given by

$$\begin{aligned}
&\epsilon_0^{N_c-\rho-n_0} \times (-1)^{N_c-\rho} q^{N_c(N_c-1)/2-\rho(S^Z-N_c)} \left(\prod_{j\in J} a_\xi^{6V}(t_j) \prod_{\ell\in\Sigma_R\setminus J} f(t_j-t_\ell) \right) \\
&\quad \times q^{-n_0(\rho+1)+(N_c-\rho)(N_c+\rho+3)/2} \frac{([N_c]_q!)^2}{[\rho]_q!} (q-q^{-1})^{N_c-\rho} \sum_{\ell=0,\ell\leqslant\rho}^{N_c-\rho-n_0} (-1)^\ell \chi^{+;J}_{\xi,N_c-\rho-n_0-\ell} \\
&\quad \times \sum_{\sigma=0}^{\rho} (-1)^\sigma q^{-(\rho-1)\sigma} q^{2\sigma(S^Z-N_c)} \sum_{J_I\cup J_K=J}^{|J_I|=\rho-\sigma, |J_K|=\sigma} \prod_{j\in J_I}\prod_{k\in J_K} f(t_k-t_j) G_\ell^{+;J_I,J_K} \\
&\quad + O\left(\epsilon_0^{N_c-\rho-n_0+1}\right).
\end{aligned} \tag{6.31}$$

Sum (6.17) over ν is now reduced into sum (6.31) over σ. Lemma 19 follows from the next lemma 20. □

Lemma 20. *Let J be a subset of Σ_R with ρ elements. We have, for generic q,*

$$\begin{aligned}
&\sum_{\sigma=0}^{\rho} (-1)^\sigma q^{-(\rho-1)\sigma} q^{2(S^Z-N_c)\sigma} \sum_{J_I\cup J_K=J}^{|J_I|=\rho-\sigma, |J_K|=\sigma} \left(\prod_{j\in J_I}\prod_{k\in J_K} f(t_k-t_j) \right) G_\ell^{+;J_I,J_K} \\
&\quad = (-1)^\rho q^{-\rho(\rho-1)/2} q^{(S^Z-N_c)\rho} (q-q^{-1})^\rho \sum_{L\subset J}^{|L|=\ell} \exp\left(\sum_{j\in L} 2t_j \right) \\
&\quad \times \prod_{i=0}^{\ell-1} [S^Z-N_c+i]_q \prod_{j=0}^{\rho-\ell-1} [S^Z-N_c-j]_q.
\end{aligned} \tag{6.32}$$

The derivation of lemma 20 is given in appendix C.

Substituting the expression of $\Sigma(\hat{\Delta}_\xi)_J$ derived in lemma 19 into proposition 16 and putting $N_c = kN$, we obtain lemma 9.

6.5. Some comments on the highest weight conjectures

The highest weight conjecture has been constructed gradually in a series of papers [14, 16–18]. It is found numerically [14] that the degenerate multiplicity of the sl_2 loop algebra should be given by some power of 2. It suggests that the degenerate eigenspace corresponds to such an irreducible representation forming a tensor product of the spin-1/2 evaluation representations. However, any connection to the Bethe ansatz was not discussed in [14]. The spectral degeneracy of the XXZ spin chain at roots of unity was carefully compared with numerical solutions of the Bethe ansatz equations in [16], and degenerate multiplets are explained in terms of complete N-strings. The first version of the highest weight conjecture was given in the last paragraph of section 3 of [17], based on numerical classification of degenerate multiplets for $L = 12$ and $N = 3$. Here, the idea of regular Bethe states is implicit but should have been known.

In [16, 17], a unique highest weight vector is assigned by such a vector that has the largest value of S^Z in a given degenerate multiplet of the sl_2 loop algebra. We can understand the highest weight conjectures of [16, 17] correctly, even without the mathematical definition of highest weight vectors.

7. Highest weight polynomials and the Drinfeld polynomials

7.1. Equivalence of the Fabricius–McCoy polynomial to the highest weight polynomial

Definition 21. *Let $|R\rangle$ be a regular Bethe state at q_0 with regular Bethe roots $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$. We define $Y_\xi(v)$ by*

$$Y_\xi(v) = \sum_{\ell=0}^{N-1} \frac{\prod_{j=1}^{L}(\sinh(v - \xi_j - (2\ell+1)\eta_0))}{\prod_{j=1}^{R} \sinh(v - \tilde{t}_j - 2\ell\eta_0) \sinh(v - \tilde{t}_j - 2(\ell+1)\eta_0)}. \tag{7.1}$$

Proposition 22. *$Y_\xi(v)$ is a Laurent polynomial of variable $z = \exp(\mp 2Nv)$ with degree $r = (L-2R)/N$ in the cases of sector A (r even) and sector B (r odd).*

Proof. First, by definition, $Y_\xi(v)$ is a rational function of variable $\exp(\mp 2v)$ with a period $2\eta_0$. In sector A where L is even, we have $\sinh(v + 2N\eta_0) = q_0^N \sinh v$ with $q_0^N = \pm 1$. In sector B where L is odd, we have $\sinh(v + 2N\eta_0) = \sinh v$ since $q_0^N = 1$. Thus, it is at least a rational function of variable $\exp(\mp 2Nv)$. Secondly, $Y_\xi(v)$ has no poles, since $\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_R$ satisfy the Bethe ansatz equations (1.4) at q_0. Thirdly, the function $Y_\xi(v)$ has the asymptotic behaviour: $Y_\xi(v) \propto \exp(\pm(L-2R)v) = z^{\mp(L-2R)/2N}$ for $v \to \pm\infty$, where $z = \exp(2Nv)$. Thus, the degree of the Laurent polynomial is given by $(L-2R)/N$. □

Lemma 23. *When p is an integer with $p \equiv 0 \pmod N$ and q_0 a root of unity with $q_0^{2N} = 1$, or when p is a half-integer with $p \equiv N/2 \pmod N$ and q_0 a primitive Nth root of unity with N odd, we have for any integer n the following:*

$$\lim_{q \to q_0} \sum_{\ell=0}^{N-1} q^{\mp(p-n)(2\ell+1)} = \begin{cases} N q_0^{\mp(p-n)} & \text{for} \quad n \equiv 0 \pmod N, \\ 0 & \text{otherwise}. \end{cases} \tag{7.2}$$

Proposition 24. *Let $|R\rangle$ be a regular Bethe state at q_0 in sector A or B in the inhomogeneous case. The Laurent polynomial $Y_\xi(v)$ of $|R\rangle$ corresponds to the highest weight polynomial $\mathcal{P}^\lambda(z)$.*

Proof. We express $Y_\xi(v)(-1)^L \, \mathrm{e}^{\mp(L-2R)v} 2^{(L-2R)}$ as follows:

$$\frac{\mathrm{e}^{\pm \sum_{j=1}^{R} 2\tilde{t}_j}}{\mathrm{e}^{\pm \sum_{k=1}^{L} 2\xi_k}} \sum_{\ell=0}^{N-1} q_0^{\mp(L/2-R)(2\ell+1)} \frac{\phi_\xi^\pm\left(\mathrm{e}^{\mp 2v} q_0^{\pm(2\ell+1)}\right)}{\tilde{F}^\pm\left(\mathrm{e}^{\mp 2v} q_0^{\pm 2\ell}\right) \tilde{F}^\pm\left(\mathrm{e}^{\mp 2v} q_0^{\pm 2(\ell+1)}\right)}. \tag{7.3}$$

We expand $Y_\xi(v)$ in terms of $\exp(\mp 2v)$. Since it is a polynomial of $\exp(\mp 2v)$, the infinite sum reduces to a finite sum with an upper bound $r = (L-2R)/N$, where the kth term vanishes at q_0 unless $k \equiv 0 \pmod N$ due to lemma 23:

$$Y_\xi(v) = \frac{\mathrm{e}^{\pm \sum_{j=1}^{R} 2\tilde{t}_j}}{2^{(L-2R)}} N q_0^{\pm(L/2-R)} z^{\mp r/2} \sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{\pm} \left(z^{\pm 1} q_0^N\right)^k. \tag{7.4}$$

Here $\tilde{\chi}_{\xi,kN}^{\pm}$ are expressed by equation (2.4) in terms of rapidities, $\tilde{t}_j$. When q_0 is a root of unity of type I, we have $\left(z q_0^N\right)^k = z^k$ for N odd $\left(q_0^N = 1\right)$, and $\left(z q_0^N\right)^k = (-z)^k$ for N even $\left(q_0^N = -1\right)$. When q_0 is a root of unity of type II, we have $\left(q_0^N z\right)^k = (-z)^k$ for N odd $\left(q_0^N = -1\right)$. We thus have for type I,

$$\sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{+} \left(z q_0^N\right)^k = \begin{cases} \sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{+} z^k & \text{for} \quad N: \text{odd} \quad \left(q_0^N = 1\right), \\ \sum_{k=0}^{r} \tilde{\chi}_{\xi,kN}^{+} (-z)^k & \text{for} \quad N: \text{even} \quad \left(q_0^N = -1\right), \end{cases} \tag{7.5}$$

and for type II,

$$\sum_{k=0}^{r} \tilde{\chi}^{+}_{\xi,kN} \left(q_0^N z\right)^k = \sum_{k=0}^{r} \tilde{\chi}^{+}_{\xi,kN} (-z)^k \qquad \left(N : \text{odd};\, q_0^N = -1\right). \tag{7.6}$$

Thus, in the cases of sectors A and B, we have shown

$$\sum_{k=0}^{r} \tilde{\chi}^{+}_{\xi,kN} \left(q_0^N z\right)^k = \mathcal{P}^{\lambda}(z). \tag{7.7}$$

□

In the homogeneous case where $\xi_n = 0$ for all n, we have the following:

Corollary 25. *For any given regular XXZ Bethe state $|R\rangle$ at q_0 in sector A or B, the Fabricius–McCoy polynomial $P^{\text{FM}}(u)$ corresponds to the highest weight polynomial $\mathcal{P}^{\lambda}(u)$.*

7.2. Examples of regular Bethe states

7.2.1. The vacuum state as a regular XXZ Bethe state. Let us calculate polynomial $\mathcal{P}^{\lambda}(u)$ for the vacuum state $|0\rangle$, where $L = 6$, $R = 0$, $N = 3$ and $q_0^3 = 1$:

$$\mathcal{P}^{\lambda}(u) = (1 - a_1 u)(1 - a_2 u) = 1 - (a_1 + a_2)u + a_1 a_2 u^2. \tag{7.8}$$

When N is odd and $q_0^N = 1$, we have $\lambda_k = (-1)^k \tilde{\chi}^{+}_{\xi,kN}$. We have $\lambda_1 = 6!/(3!)^2 = 20$, $\lambda_2 = 6!/(6!0!) = 1$. The highest weight parameters are thus given by

$$\hat{a}_1, \hat{a}_2 = 10 \pm 3\sqrt{11}. \tag{7.9}$$

Since $\hat{a}_1$ and $\hat{a}_2$ are distinct ($m_1 = m_2 = 1$), the vacuum state $|0\rangle$ generates an irreducible representation. We thus have $(1 + 1)^2 = 4$ as the dimension.

Let us discuss the case where $L = 6$, $R = 0$, $N = 3$ and $q_0^3 = -1$. When N is odd and $q_0^N = -1$, we have $\lambda_k = \tilde{\chi}^{+}_{\xi,kN}$. We thus have

$$\hat{a}_1, \hat{a}_2 = -10 \pm 3\sqrt{11}. \tag{7.10}$$

For the XXZ Hamiltonian (1.1) with L even, q is mapped to $-q$ by the unitary transformation, $\prod_{j=1}^{L/2} \sigma_{2j}^{Z}$. Here, the Hamiltonian $\mathcal{H}_{\text{XXZ}}$ is mapped to $-\mathcal{H}_{\text{XXZ}}$, and highest weight parameters $\hat{a}_j$ are transformed to $-\hat{a}_j$ for $j = 1, 2$.

7.2.2. The regular XXZ Bethe state with one down-spin. We now discuss the case of $L = 8$, $R = 1$, $N = 3$ and $q_0^3 = 1$. Here $q_0 = \exp(\pm 2\pi\sqrt{-1}/3)$. Let us specify the Bethe root as $\exp(2t_2) = (1 - \sqrt{-1}q)/(q_0 - \sqrt{-1})$. Noting $[3\ell + 1]_{q_0} = 1$, $[3\ell + 2]_{q_0} = -1$ and $[3\ell]_{q_0} = 0$ for integers ℓ, we derive $\tilde{\chi}_3^{+} = -13(2 - \sqrt{3})$ and $\tilde{\chi}_6^{+} = 7 - 4\sqrt{3}$. We thus have

$$\mathcal{P}^{\lambda}(u) = 1 - 13(2 - \sqrt{3})u + (7 - 4\sqrt{3})u^2, \tag{7.11}$$

where highest weight parameters $\hat{a}_1$ and $\hat{a}_2$ are given by

$$\hat{a}_1, \hat{a}_2 = \tfrac{1}{2}(13 \pm \sqrt{165})(2 - \sqrt{3}). \tag{7.12}$$

7.2.3. The case of a reducible Weyl module: the inhomogeneous case with degenerate evaluation parameters. We now discuss such a regular Bethe state that has degenerate highest weight parameters. Let us consider the inhomogeneous case of $L = 6$, $R = 0$, $N = 3$ and $q_0 = \exp(2\pi i/3)$. We set $\xi_1 \neq 0$, while $\xi_2 = \xi_3 = \xi_4 = \xi_5 = \xi_6 = 0$. We have $\phi_\xi^+(x) = (1 - yx)(1 - x)^5$, where $y = \exp(2\xi_1)$. Expanding $\phi_\xi^+(x)$, we have $\chi_0^+ = 1$, $\chi_3^+ = -5!/(3!2!) - 5!/(2!3!)y = -(10 + 10y)$ and $\chi_6^+ = y$. We have $\mathcal{P}^\lambda(u) = 1 - 10(1 + y)u + yu^2$, where parameters $\hat{a}_1$ and $\hat{a}_2$ are given by

$$\hat{a}_1, \hat{a}_2 = 5\big(1 + y \pm \sqrt{(1 + y)^2 - y/5}\big). \tag{7.13}$$

We have $\hat{a}_1 = \hat{a}_2$ if and only if $y = y_c = (-49 \pm 3\sqrt{-11})/50$.

When $y \neq y_c$, $\hat{a}_1$ and $\hat{a}_2$ are distinct, and hence $U|0\rangle$ is irreducible. We have $\dim U|0\rangle = (1 + 1)^2 = 4$, and $\mathcal{P}^\lambda(u)$ gives the Drinfeld polynomial.

When $y = y_c$, however, $\hat{a}_1$ and $\hat{a}_2$ are degenerate:

$$\hat{a}_1 = \hat{a}_2 = a_1 = \tfrac{1}{10}(1 \pm 3\sqrt{-11}). \tag{7.14}$$

Recall that $U|0\rangle$ is irreducible if and only if $\left(x_1^- - a_1 x_0^-\right)|0\rangle$ vanishes. We apply (2.10) with $r = 2$, where generators x_1^- and x_0^- are given by $T_\xi^{-(3)} = V^- T^{-(3)} V^+$ and $S_\xi^{-(3)} = V^+ S^{-(3)} V^-$, respectively. Here we recall (4.15). The (1, 1, 1, 2, 2, 2) elements of vectors $T_\xi^{-(N)}|0\rangle$ and $S_\xi^{-(N)}|0\rangle$ are given by $q_0^{3/2}$ and $q_0^{-3/2}$, respectively, and hence we have $T_\xi^{-(N)}|0\rangle \neq a_1 S_\xi^{-(N)}|0\rangle$. We thus conclude that $U|0\rangle$ is reducible. We now derive $\dim U|0\rangle = 4$, and hence $U|0\rangle$ gives a Weyl module. In fact, the basis is given by $|0\rangle$, $x_0^-|0\rangle$, $\left(x_0^-\right)^2|0\rangle$ and $w = \left(x_1^- - a_1 x_0^-\right)|0\rangle$. We also confirm that it is reducible, noting that $x_k^+ w = 0$ for $k \in \mathbf{Z}$.

8. Concluding remark

The highest weight conjecture has been shown in sectors A and B in the paper. However, operators commuting with the transfer matrix are constructed also in other sectors [14]. We thus have a conjecture that in any sector of S^Z, some version of regularized Bethe ansatz eigenvectors should be highest weight (see also [18]).

Acknowledgments

The author would like to thank Professors M Jimbo and B M McCoy for helpful comments. He would also like to thank Professor K Fabricius for useful comments. He is grateful to Professor A Nakayashiki for useful comments during the RIMS conference 'Solvable Lattice Models 2004' in 20–23 July 2004. This work is partially supported by the grant-in-aid for Young Scientists (A): no. 14702012.

Appendix A. Derivation of lemma 6 for showing highest weight properties

For a given integer ℓ, let $U(\mathcal{B}_\ell)$ be such a subalgebra of $U(L(sl_2))$ that is generated by h_k, $x_{\ell+k}^+$ and $x_{-\ell+1+k}^-$ for $k \in \mathbf{Z}_{\geqslant 0}$. We denote by $\mathcal{B}_\ell^+$ such a subalgebra of $U(\mathcal{B}_\ell)$ that is generated by $x_{\ell+k}^+$ for $k \in \mathbf{Z}_{\geqslant 0}$. We express by $(X)^{(n)}$ the nth power of X divided by the factorial of n, i.e. $(X)^{(n)} = (X)_{q=1}^{(n)} = X^n/n!$.

Lemma A.1. *Let ℓ be an integer. For a given positive integer n we have*

$$(\mathrm{A}_n) : \left(x_\ell^+\right)^{(n-1)}\left(x_{1-\ell}^-\right)^{(n)} = \sum_{k=1}^{n}(-1)^{k-1}x_{k-\ell}^-\left(x_\ell^+\right)^{(n-k)}\left(x_{1-\ell}^-\right)^{(n-k)} \bmod U(\mathcal{B}_\ell)\mathcal{B}_\ell^+,$$

$$(\mathrm{B}_n) : \left(x_\ell^+\right)^{(n)}\left(x_{1-\ell}^-\right)^{(n)} = \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}h_k\left(x_\ell^+\right)^{(n-k)}\left(x_{1-\ell}^-\right)^{(n-k)} \bmod U(\mathcal{B}_\ell)\mathcal{B}_\ell^+,$$

$$(\mathrm{C}_n) : \left[h_j, \left(x_\ell^+\right)^{(m)}\left(x_{1-\ell}^-\right)^{(m)}\right] = 0 \bmod U(\mathcal{B}_\ell)\mathcal{B}_\ell^+ \qquad \text{for} \quad m \leqslant n \text{ and } j \in \mathbf{Z}.$$

Proof. We first show the following relations by induction on n:

$$\left[h_1, \left(x_\ell^+\right)^{(n)}\right] = 2x_{\ell+1}^+\left(x_\ell^+\right)^{(n-1)}, \qquad \left[\left(x_\ell^+\right)^{(n)}, x_{1-\ell}^-\right] = \left(x_\ell^+\right)^{(n-1)}h_1 + x_{1+\ell}^+\left(x_\ell^+\right)^{(n-2)}, \tag{A.1}$$

$$\left[h_1, \left(x_{1-\ell}^-\right)^{(n)}\right] = (-2)x_{2-\ell}^-\left(x_{1-\ell}^-\right)^{(n-1)}, \qquad \left[x_\ell^+, \left(x_{1-\ell}^-\right)^{(n)}\right] = \left(x_1^-\right)^{(n-1)}h_1 - x_{2-\ell}^-\left(x_{1-\ell}^-\right)^{(n-2)}. \tag{A.2}$$

Through the recursive relations, we derive the following inductive formula for the product $\left(x_\ell^+\right)^{(n-1)}\left(x_{1-\ell}^-\right)^{(n)}$ with respect to n:

$$\left(x_\ell^+\right)^{(n)}\left(x_{1-\ell}^-\right)^{(n+1)} = x_{1-\ell}^-\left(x_\ell^+\right)^{(n)}\left(x_{1-\ell}^-\right)^{(n)} + \tfrac{1}{2}\left[h_1, \left(x_\ell^+\right)^{(n-1)}\left(x_{1-\ell}^-\right)^{(n)}\right]$$
$$- \left(x_\ell^+\right)^{(n-1)}\left(x_{1-\ell}^-\right)^{(n+1)}x_\ell^+, \qquad \text{for} \quad \ell \in \mathbf{Z}. \tag{A.3}$$

We now show three relations (A_n), (B_n) and (C_n), inductively on n as follows. We show $(\mathrm{A}_1), (\mathrm{A}_2), (\mathrm{B}_1)$ and (C_1), directly. Relation (A_n) is derived from (A_{n-1}) and (C_{n-2}). Here we make use of formula (A.3). We derive relation (B_n) from $(\mathrm{A}_n), (\mathrm{C}_{n-1})$ and (B_m) for $m \leqslant n-1$, multiplying both hand sides of (A_n) by x_ℓ^+ from the left. We show $x_\ell^+\left(x_\ell^+\right)^{(m)}\left(x_{1-\ell}^-\right)^{(m)} \in U(\mathcal{B}_\ell)\mathcal{B}_\ell^+$ for $m \leqslant n-1$ by induction on m. Here we make use of (B_m) for $m \leqslant n-1$ and (C_{n-1}). Finally, (C_n) is derived from (B_{n-1}) and (C_{n-1}). Thus, the cycle of induction process, (A_n), (B_n) and (C_n), is closed. □

We remark that relation (A.3) for $\ell = 0$ is given by the classical limit of formula $(\mathrm{iv})_r$ in [6, subsection 3.5]. It is also reviewed in [13, appendix A].

Lemma A.2. *Let ℓ be an integer. If a vector Ψ in a finite-dimensional representation of $U(\mathcal{B}_\ell)$ satisfies*

$$\begin{aligned} &x_{\ell+k}^+\Psi = 0, \qquad && h_k\Psi = d_k\Psi \qquad \textit{for} \quad k \in \mathbf{Z}_{\geqslant 0}, \\ &\left(x_{1-\ell}^-\right)^r\Psi \neq 0, \qquad && \left(x_{1-\ell}^-\right)^{r+1}\Psi = 0, \end{aligned} \tag{A.4}$$

then we have the following. (i) For a given non-negative integer $n \leqslant r$ and a given set of integers k_j satisfying $1-\ell \leqslant k_1 \leqslant \cdots \leqslant k_n$, we have with some $A_{k_1,\ldots,k_n} \in \mathbf{C}$:

$$\left(x_{1-\ell}^-\right)^{r-n}x_{k_1}^-\cdots x_{k_n}^-\Psi = A_{k_1,\ldots,k_n}\left(x_{1-\ell}^-\right)^r\Psi. \tag{A.5}$$

(ii) The subspace of weight $-r$ of $U(\mathcal{B}_\ell)\Psi$ is one dimensional. Here we call eigenvalues of h_0 weights. (iii) Recall that eigenvalues λ_n are defined by (5.3) i.e. (A.6) for $\ell = 0$. Then, we have

$$\left(x_\ell^+\right)^{(n)}\left(x_{1-\ell}^-\right)^{(n)}\Psi = \lambda_n\Psi, \qquad \textit{for} \quad n = 1, 2, \ldots, r. \tag{A.6}$$

Proof. We show (i) by induction on n. The case of $n = 0$ is trivial. Let us assume the cases of $n-1$ and n. We have

$$x_m^+\left(x_{1-\ell}^-\right)^{r+1-n}x_{k_1}^-\cdots x_{k_n}^-\Psi = A_{k_1,\ldots,k_n}x_m^+\cdot\left(x_{1-\ell}^-\right)^{r+1}\Psi = 0.$$

Calculating $\left[x_m^+, \left(x_{1-\ell}^-\right)^{(r+1-n)} x_{k_1}^- \cdots x_{k_n}^-\right]$, we derive (A.5) in the case of $n+1$. We now show (ii). Applying the Poincare–Birkhoff–Witt theorem to $U(\mathcal{B}_\ell)$, we have that every vector v in the subspace of weight $-r$ of $U\Psi$ is expressed as a linear combination of monomial vectors $x_{k_1}^- \cdots x_{k_r}^- \Psi$ for sets of integers k_j satisfying $1-\ell \leqslant k_1 \leqslant \cdots \leqslant k_r$ [13]. Thus, we obtain (ii) from (i). We show (iii) by induction on n. We derive the $n=1$ case by $\left[x_\ell^+, x_{1-\ell}^-\right] = h_1$. Let us assume (A.6) for the cases of $1, 2, \ldots, n-1$. We derive the case of n through (B_n) of lemma A.1. □

Let $\mathcal{U}_k$ be the sl_2 subalgebra generated by x_{-k}^+, x_k^- and h_0 for an integer k.

Proof of lemma 6. We show it in sequence as follows: (i) $h_1|\Phi\rangle = d_1|\Phi\rangle$, where $d_1 \in \boldsymbol{C}$; (ii) $x_k^+|\Phi\rangle = 0$ ($k \in \boldsymbol{Z}_{>0}$); (iii) $h_k|\Phi\rangle = d_k|\Phi\rangle$ ($k \in \boldsymbol{Z}_{>0}$), where $d_k \in \mathbf{C}$; (vi) $h_{-1}|\Phi\rangle = d_{-1}|\Phi\rangle$, where $d_{-1} \in \boldsymbol{C}$; (v) $x_{-k}^+|\Phi\rangle = 0$ ($k \in \boldsymbol{Z}_{>0}$); (vi) $h_{-k}|\Phi\rangle = d_{-k}|\Phi\rangle (k \in \boldsymbol{Z}_{>0})$, where $d_{-k} \in \boldsymbol{C}$. Let us first note that for $k=0$ and 1, $\mathcal{U}_k|\Phi\rangle$ corresponds to the $(r+1)$-dimensional irreducible representation of $U(sl_2)$. We therefore have $\left(x_k^-\right)^r|\Phi\rangle \neq 0$ and $\left(x_k^-\right)^{r+1}|\Phi\rangle = 0$ for $k = 0, 1$. We derive (i) from (5.1) and (5.3), noting $h_1 = \left[x_0^+, x_1^-\right]$. We show (ii) through induction on k:

$$\begin{aligned} x_{k+1}^+|\Phi\rangle &= \tfrac{1}{2}\left(h_1 x_k^+ - x_k^+ h_1\right)|\Phi\rangle \\ &= \tfrac{1}{2}(h_1 - d_1)x_k^+|\Phi\rangle, \qquad \text{for} \quad k \in \boldsymbol{Z}_{\geqslant 0}. \end{aligned} \tag{A.7}$$

We derive (iii) by induction on k using (B_k) of lemma A.1 for $\ell = 0$:

$$h_k|\Phi\rangle = \sum_{j=1}^{k-1} (-1)^{k-j+1} \lambda_{k-j} h_j |\Phi\rangle + k(-1)^{k-1}\lambda_k|\Phi\rangle. \tag{A.8}$$

In order to derive (iv), we first show that $\lambda_r \neq 0$ as follows. We note that vector $|\Phi\rangle$ satisfies the necessary conditions of Ψ in lemma A.2, since we have shown (ii) and (iii) in the above. From (iii) of lemma A.2 for $\ell = 1$, we have

$$\left(x_1^+\right)^{(n)}\left(x_0^-\right)^{(n)}|\Phi\rangle = \lambda_n^+|\Phi\rangle \qquad \text{for} \quad n = 1, 2, \ldots, r. \tag{A.9}$$

From (ii) of lemma A.2, we have $\left(x_1^-\right)^r|\Phi\rangle = A_1\left(x_0^-\right)^r|\Phi\rangle$. Here $A_1 \neq 0$, since $\left(x_1^-\right)^r|\Phi\rangle \neq 0$ due to $\mathcal{U}_1$. We thus obtain that $\lambda_r \neq 0$ [13]. We then consider (A_{r+1}) of lemma A.1 for $\ell = 1$:

$$\left(x_1^+\right)^{(r)}\left(x_0^-\right)^{(r+1)} = \sum_{k=1}^{r+1} (-1)^{k-1} x_{k-1}^- \left(x_1^+\right)^{(r+1-k)}\left(x_0^-\right)^{(r+1-k)} \bmod U(\mathcal{B}_1)\mathcal{B}_1^+. \tag{A.10}$$

Introducing $\lambda_0 = 1$, we have from (A.10),

$$\sum_{j=0}^{r} (-1)^j \lambda_{r-j} x_j^- |\Phi\rangle = 0. \tag{A.11}$$

Applying x_{-1}^+ to (A.11), we have

$$\lambda_r h_{-1}|\Phi\rangle = \sum_{j=1}^{r} (-1)^{j-1} \lambda_{r-j} h_{j-1}|\Phi\rangle. \tag{A.12}$$

Thus, we obtain $h_{-1}|\Phi\rangle = d_{-1}|\Phi\rangle$, where d_{-1} is defined by

$$d_{-1} = \frac{1}{\lambda_r} \sum_{j=1}^{r} (-1)^{j-1} \lambda_{r-j} d_{j-1}. \tag{A.13}$$

We show (v) inductively with respect to k:

$$
\begin{aligned}
x^{+}_{-(k+1)}|\Phi\rangle &= \tfrac{1}{2}\left(h_{-1}x^{+}_{-k} - x^{+}_{-k}h_{-1}\right)|\Phi\rangle \\
&= \tfrac{1}{2}(h_{-1} - d_{-1})x^{+}_{-k}|\Phi\rangle \qquad \text{for} \quad k \in \boldsymbol{Z}_{\geqslant 0}.
\end{aligned} \tag{A.14}
$$

Multiplying (A.11) with x^{+}_{-k}, we show (vi) through induction on k by the following:

$$
h_{-k}|\Phi\rangle = \frac{1}{\lambda_r}\sum_{j=1}^{r}(-1)^{j-1}\lambda_{r-j}h_{j-k}|\Phi\rangle. \tag{A.15}
$$

□

Appendix B. Some combinatorial formulas

Lemma B.1 (q-binomial theorem)**.** *For a positive integer n, we have*

$$
\prod_{\ell=0}^{n-1}(1 - q^{\pm 2\ell}z) = \sum_{j=0}^{n}(-1)^j q^{\pm(n-1)j} z^j \begin{bmatrix} n \\ j \end{bmatrix}_q . \tag{B.1}
$$

Here q and z are arbitrary.

Lemma B.2. *Let w_j be arbitrary parameters for $j = 1, 2, \ldots, n$. For an integer $n > 0$, we have the following:*

$$
\sum_{P\in\mathcal{S}(n)}\prod_{1\leqslant j<k\leqslant n} f(w_{Pj} - w_{Pk}) = \sum_{P\in\mathcal{S}(n)}\prod_{1\leqslant j<k\leqslant n} f(w_{Pk} - w_{Pj}) = [n]_q!. \tag{B.2}
$$

Proof. We express the sum on the left-hand side of equation (B.2) as $F(w_1, \ldots, w_n)$. Let us take an integer j satisfying $1 \leqslant j \leqslant n$. As a function of variable w_j the quantity $F(w_1, \ldots, w_n)$ is a meromorphic function with no poles, and it is bounded at infinity, $w_j = \infty$. Therefore it is given by a constant with respect to the variable w_j. Similarly, we show that the quantity $F(w_1, \ldots, w_n)$ is a constant with respect to all variables w_j for $j = 1, 2, \ldots, n$. Let us evaluate the constant by substituting w_j with z_j of the complete n-string (4.17). The summand of the sum on the left-hand side of (B.2) vanishes except for such a permutation P that gives $(P1, P2, \ldots, Pn) = (n, n-1, \ldots, 1)$. Thus, we have the equality (B.2). □

Lemma B.3. *Let m and n be integers satisfying $m \geqslant n \geqslant 0$, and w_j for $1 \leqslant j \leqslant m$ be arbitrary parameters. Then, we have*

$$
\sum_{S_n\subseteq\Sigma_m}^{|S_n|=n}\prod_{j\in\Sigma_m\setminus S_n}\prod_{k\in S_n} f(w_j - w_k) = \sum_{S_n\subseteq\Sigma_m}^{|S_n|=n}\prod_{j\in\Sigma_m\setminus S_n}\prod_{k\in S_n} f(w_k - w_j) = \begin{bmatrix} m \\ n \end{bmatrix}_q . \tag{B.3}
$$

Here $\Sigma_m = \{1, 2, \ldots, m\}$, and the sum is taken over all such subsets S_n of Σ_m that have n elements. Here $\prod_{j\in A}\prod_{k\in B} f(w_j - w_k) = 1$ if A or B is empty.

Proof. We express the sum on the left-hand side of equation (B.3) as $G(w_1, \ldots, w_m)$. Let us take an integer j satisfying $1 \leqslant j \leqslant m$. As a function of variable w_j the quantity $G(w_1, \ldots, w_m)$ is a meromorphic function with no poles, and it is bounded at infinity, $w_j = \infty$. Therefore it is a constant with respect to w_j. Similarly, we show that $G(w_1, \ldots, w_m)$ is a constant with respect to all variables w_j for $j = 1, 2, \ldots, m$. Let us evaluate the constant by substituting w_j with z_j of the complete m-string (4.17). The product $\prod_{j\in\Sigma_m\setminus S_n}\prod_{k\in S_n} f(w_j - w_k)$ vanishes except for the case of $S_n = \{1, 2, \ldots, n\}$. We thus have the equality (B.3). □

Appendix C. Proof of lemma 14 by induction on n

The case of $n = 1$ is trivial. Suppose that the case of $n - 1$ holds. We denote by $\mathcal{C}_n$ the cyclic group generated by a cyclic permutation $(1, 2, \ldots, n)$, and by $\mathcal{T}_{n-1}$ the symmetric group on the set $\{2, 3, \ldots, n\}$. Let us remark that any element P of $\mathcal{S}(n)$ is given by a product of an element σ of $\mathcal{C}_n$ and an element τ of $\mathcal{T}_{n-1}$. Putting $P = \sigma\tau$ in expression (6.5), we have $\Delta^{\pm}_{S_n;\Sigma_M}$ as

$$\sum_{\sigma\in\mathcal{C}_n}\sum_{\tau\in\mathcal{T}_{n-1}}\prod_{\ell=1}^{n}\left(\alpha_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{\sigma\tau\ell}}\right)\prod_{k=\ell+1}^{n}q^{\pm 1}f\left(w_{j_{\sigma\tau\ell}}-w_{j_{\sigma\tau k}}\right)\right.$$
$$\left.-\bar{\alpha}_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{\sigma\tau\ell}}\right)\prod_{k=\ell+1}^{n}q^{\mp 1}f\left(w_{j_{\sigma\tau k}}-w_{j_{\sigma\tau\ell}}\right)\right). \tag{C.1}$$

Here we note that $\sigma\tau 1 = \sigma 1$. For the case of $\ell = 1$, we have

$$\prod_{k=2}^{n}f\left(w_{j_{\sigma\tau 1}}-w_{j_{\sigma\tau k}}\right)=\prod_{k=2}^{n}f\left(w_{j_{\sigma 1}}-w_{j_{\sigma\tau k}}\right)=\prod_{k=2}^{n}f\left(w_{j_{\sigma 1}}-w_{j_{\sigma k}}\right), \tag{C.2}$$

and we have $\Delta^{\pm}_{S_n;\Sigma_M}$ as follows:

$$\sum_{\sigma\in\mathcal{C}_n}\left\{\left(\alpha_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{\sigma 1}}\right)\prod_{k=2}^{n}q^{\pm 1}f\left(w_{j_{\sigma 1}}-w_{j_{\sigma k}}\right)-\bar{\alpha}_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{\sigma 1}}\right)\prod_{k=2}^{n}q^{\mp 1}f\left(w_{j_{\sigma k}}-w_{j_{\sigma 1}}\right)\right)\right.$$
$$\times\sum_{\tau\in\mathcal{T}_{n-1}}\prod_{\ell=2}^{n}\left(\alpha_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{\sigma\tau\ell}}\right)\prod_{k=\ell+1}^{n}q^{\pm 1}f\left(w_{j_{\sigma\tau\ell}}-w_{j_{\sigma\tau k}}\right)\right.$$
$$\left.\left.-\bar{\alpha}_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{j_{\sigma\tau\ell}}\right)\prod_{k=\ell+1}^{n}q^{\mp 1}f\left(w_{j_{\sigma\tau k}}-w_{j_{\sigma\tau\ell}}\right)\right)\right\}. \tag{C.3}$$

Here we note that $\mathcal{T}_{n-1}$ is equivalent to $\mathcal{S}(n-1)$. Let us define j_σ by $(j_\sigma)_k = j_{\sigma k}$ $(k = 1, \ldots, n)$. Applying the induction assumption to the case where S_{n-1} is given by $\{j_{\sigma k}|k = 2, \ldots, n\}$, the last line of (C.3) is given by

$$\sum_{\tau\in\mathcal{T}_{n-1}}\prod_{\ell=2}^{n}\left(\alpha_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{(j_\sigma)_{\tau\ell}}\right)\prod_{k=\ell+1}^{n}q^{\pm 1}f\left(w_{(j_\sigma)_{\tau\ell}}-w_{(j_\sigma)_{\tau k}}\right)\right.$$
$$\left.-\bar{\alpha}_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{(j_\sigma)_{\tau\ell}}\right)\prod_{k=\ell+1}^{n}q^{\mp 1}f\left(w_{(j_\sigma)_{\tau k}}-w_{(j_\sigma)_{\tau\ell}}\right)\right)$$
$$=\sum_{\tau\in\mathcal{T}_{n-1}}\sum_{k=0}^{n-1}(-1)^k\begin{bmatrix}n-1\\k\end{bmatrix}_q q^{\pm(n-1)(n-2)/2}q^{\mp(n-2)k}$$
$$\times\prod_{2\leqslant\ell\leqslant n-k}\alpha_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{(j_\sigma)_{\tau\ell}}\right)\prod_{n-k<\ell\leqslant n}\bar{\alpha}_{\xi}^{\pm;\Sigma_M\setminus S_n}\left(w_{(j_\sigma)_{\tau\ell}}\right)$$
$$\times\prod_{2\leqslant\ell<m\leqslant n}f\left(w_{(j_\sigma)_{\tau\ell}}-w_{(j_\sigma)_{\tau m}}\right). \tag{C.4}$$

Substituting it to (C.3), rewriting $\sigma\tau$ as $P \in \mathcal{S}(n)$, and using the recursive relations of the q-binomial coefficients, we obtain formula (6.9) for the case of n.

Appendix D. Proof of lemma 15

We call a sequence of integers given by $n_k = n_1 + k - 1$ for $k = 1, \ldots, \ell$ with positive integers n_1 and ℓ, a slope 1 increasing sequence of integers. Let us assume that $Z_{N_c} \backslash W$ is given by the union of sets of slope 1 increasing sequences of integers, X_j: $Z_{N_c} \backslash W = X_1 \cup X_2 \cup \cdots \cup X_m$. We have $Z_{N_c} = Y_0 \cup X_1 \cup Y_1 \cdots \cup X_m \cup Y_m$, where Y_j are slope 1 increasing sequences of integers, and $W = Y_0 \cup Y_1 \cup \cdots \cup Y_m$. Recall the notation $\underline{s} = s + R$. We now show that if Y_1 is given by $Y_1 = \{\underline{s}, \underline{s}+1, \ldots, \underline{t}\}$ with $s \leqslant t$, we have

$$\Delta(\xi)^{\pm; \ldots, s-1, t+1, \ldots}_{j_1, \ldots, j_\rho} = 0. \tag{D.1}$$

Let us first consider the case of $s < t$. We express elements of S_{N_c} as $i_1, i_2, \ldots, i_{N_c}$. Recall that each permutation P gives sequence $\left(i_{P1}, i_{P2}, \ldots, i_{PN_c}\right)$. From the fact

$$f(z_s - z_{s+1}) = \cdots = f(z_{t-1} - z_t) = 0, \tag{D.2}$$

it follows that product $\prod_{1 \leqslant \ell < m \leqslant N_c} f\left(w_{i_{P\ell}} - w_{i_{Pm}}\right)$ in formula (6.9) vanishes unless integers $\underline{s}, \underline{s}+1, \ldots, \underline{t}$ appear in reverse order in sequence $\left(i_{P1}, i_{P2}, \ldots, i_{PN_c}\right)$: if $\underline{a} = i_{Pj(a)}$ and $\underline{a}+1 = i_{Pj(a+1)}$ then $j(a+1) < j(a)$ for $s \leqslant a \leqslant t$, i.e. $\underline{s}$ comes later than $\underline{s}+1$ in $\left(i_{P1}, i_{P2}, \ldots, i_{PN_c}\right)$, and so on. We next consider product $\prod_{1 \leqslant \ell \leqslant N_c - k} \alpha_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}\left(w_{i_{P\ell}}\right)$. If subsequence $\left(i_{P1}, i_{P2}, \ldots, i_{P(N_c-k)}\right)$ contains integer $\underline{t}$, then the product vanishes. Similarly, if subsequence $\left(i_{P(N_c-k+1)}, \ldots, i_{P(N_c-1)}, i_{PN_c}\right)$ contains integer $\underline{s}$, then product $\prod_{N_c-k < \ell \leqslant N_c} \bar{\alpha}_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}\left(w_{i_{P\ell}}\right)$ vanishes. In order to make the products nonzero, integer $\underline{s}$ should be contained in subsequence $\left(i_{P1}, i_{P2}, \ldots, i_{P(N_c-k)}\right)$ and integer $\underline{t}$ in $\left(i_{P(N_c-k+1)}, \ldots, i_{P(N_c-1)}, i_{PN_c}\right)$. However, it is not compatible with the constraint that integers $\underline{s}, \underline{s}+1, \ldots, \underline{t}$ should appear in reverse order in sequence $\left(i_{P1}, i_{P2}, \ldots, i_{PN_c}\right)$. Thus, the summand of the sum $\Delta(\xi)^{\pm}_{S_{N_c}; \Sigma_{R+N_c}}$ in (6.9) vanishes for any permutation P, and hence the sum $\Delta^{\pm}_{S_{N_c}; \Sigma_{R+N_c}}$ vanishes. In the case of $s = t$, we show that $\prod_{1 \leqslant \ell \leqslant N_c - k} \alpha_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}(w_{i_{P\ell}}) = 0$ or $\prod_{N_c-k < \ell \leqslant N_c} \bar{\alpha}_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}\left(w_{i_{P\ell}}\right) = 0$ for any k and P.

For an illustration, we show $\Delta(\xi)^{\pm; 1,2,5}_{j_1, j_2, j_3} = 0$ for $N_c = 5$. Since $f(z_3 - z_4) = 0$, product $\prod_{1 \leqslant \ell < m \leqslant 5} f\left(w_{i_{P\ell}} - w_{i_{Pm}}\right)$ vanishes unless $\underline{4}$ comes earlier than $\underline{3}$ in $(i_{P1}, i_{P2}, i_{P3}, i_{P4}, i_{P5})$. When $Z_{N_c} \backslash W = \{\underline{1}, \underline{2}, \underline{5}\}$, we have $\bar{\alpha}_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}(z_3) = 0$ and $\alpha_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}(z_4) = 0$. If $\underline{4} \in \left\{i_{P1}, i_{P2}, \ldots, i_{P(N_c-k)}\right\}$, product $\prod_{1 \leqslant \ell \leqslant N_c - k} \alpha_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}\left(w_{j_{P\ell}}\right)$ vanishes. If $\underline{3} \in \left\{i_{P(N_c-k+1)}, \ldots, i_{PN_c}\right\}$, then product $\prod_{N_c-k < \ell \leqslant N_c} \bar{\alpha}_\xi^{\pm; \Sigma_{R+N_c} \backslash S_{N_c}}\left(w_{j_{P\ell}}\right)$ vanishes. However, it is not compatible with the constraint that $\underline{4}$ comes earlier than $\underline{3}$ in sequence $(i_{P1}, i_{P2}, \ldots, i_{P5})$. Therefore, we have $\Delta(\xi)^{\pm; 1,2,5}_{j_1, j_2, j_3} = 0$.

Appendix E. Derivation of equation (6.32)

Generalizing $G_\ell^{+; J_I, J_K}$ given by equation (6.28), we define for $\ell \leqslant r$ the following:

$$C_\ell^{J_I, J_K}(m) = \sum_{\ell_I = 0}^{\ell} q^{m(\ell_K - \ell_I)} \sum_{L_I \subset J_I}^{|L_I| = \ell_I} \exp\left(\sum_{j \in L_I} 2t_j\right) \sum_{L_K \subset J_K}^{|L_K| = \ell_K} \exp\left(\sum_{j \in L_K} 2t_j\right), \tag{E.1}$$

where $\ell_K = \ell - \ell_I$. When $m = \rho - 1$, the generalized coefficient gives the original one: $C_\ell^{J_I, J_K}(\rho - 1) = G_\ell^{+; J_I, J_K}$. In terms of $C_\ell^{J_I, J_K}(m)$, we introduce

$$J(\rho, \sigma; \ell)_m = \sum_{J_I \cup J_K = J}^{|J_K| = \rho - \sigma, |J_K| = \sigma} \left(\prod_{j \in J_I} \prod_{k \in J_K} f(w_k - w_j)\right) C_\ell^{J_I, J_K}(m), \tag{E.2}$$

Here, w_j $(1 \leqslant j \leqslant r)$ are arbitrary parameters, and the sum is taken over all pairs of disjoint subsets J_I and J_K of J with $J_I \cup J_K = J$. We note that $J(\rho, \sigma; \ell)_m$ vanishes when $\ell > \rho$ or $\sigma > \rho$ by definition. Let us denote by $\Sigma(J)^{\rho}_{\ell,m}$ the following sum over σ:

$$\Sigma(J)^{\rho}_{\ell,m} = \sum_{\sigma=0}^{\rho} (-1)^{\sigma} q^{-(\rho-1)\sigma} x^{\sigma} J(\rho, \sigma; \ell)_m. \tag{E.3}$$

The main result of appendix E is given as follows:

$$\Sigma(J)^{r}_{\ell,m} = q^{-\ell m} \prod_{i=1}^{\ell} (1 - x q^{2m-2(\rho-i)}) \prod_{j=0}^{\rho-\ell-1} (1 - x q^{-2j}) \sum_{L \subset J}^{|L|=\ell} \exp\left(\sum_{j\in L} 2t_j\right). \tag{E.4}$$

Let us derive (E.4). We first show that $J(\rho, \sigma; \ell)_m$ is expressed as follows:

$$J(\rho, \sigma; \ell)_m = \sum_{t=0}^{\ell} \begin{bmatrix} \rho - \ell \\ \sigma - t \end{bmatrix}_q \begin{bmatrix} \ell \\ t \end{bmatrix}_q q^{\ell(\sigma-m)+(2m-\rho)t} \sum_{L\subset J}^{|L|=\ell} \exp\left(\sum_{j\in L} 2t_j\right). \tag{E.5}$$

Expression (E.5) is derived through induction on ℓ. The case of $\ell = 0$ is given by formula (B.3). For the case of $\ell > 0$, we note that we have from (E.1)

$$\lim_{t_{j_\rho}\to\infty} J(\rho, \sigma; \ell)_m / \exp(2t_{j_\rho}) = J(\rho - 1, \sigma; \ell - 1)_m q^{\sigma-m} + J(\rho - 1, \sigma - 1; \ell - 1)_m q^{-\rho+\sigma+m}, \tag{E.6}$$

and also that $J(\rho, \sigma; \ell)_m$ is symmetric with respect to $\exp 2t_{j_1}, \exp 2t_{j_2}, \ldots, \exp 2t_{j_\rho}$, by definition. Making use of (E.6) and the symmetric property, we have (E.5). We now evaluate coefficient $Z(\rho, \sigma; \ell)_m$ defined in the following:

$$J(\rho, \sigma; \ell)_m = Z(\rho, \sigma; \ell)_m \sum_{L\subset J}^{|L|=\ell} \exp\left(\sum_{j\in L} 2t_j\right). \tag{E.7}$$

From the recursion relation (E.6), we have

$$Z(\rho, \sigma; \ell)_m = Z(\rho - 1, \sigma; \ell - 1)_m q^{\sigma-m} + Z(\rho - 1, \sigma - 1; \ell - 1)_m q^{-\rho+\sigma+m}. \tag{E.8}$$

Let us define $\Sigma(Z)^{\rho}_{\ell,m}$ by

$$\Sigma(Z)^{\rho}_{\ell,m} = \sum_{\sigma=0}^{\rho} (-1)^{\sigma} q^{-(\rho-1)\sigma} x^{\sigma} Z(\rho, \sigma; \ell)_m. \tag{E.9}$$

We reformulate the sum over σ as follows:

$$\Sigma(J)^{\rho}_{\ell,m} = \Sigma(Z)^{\rho}_{\ell,m} \sum_{L\subset J}^{|L|=\ell} \exp\left(\sum_{j\in L} 2t_j\right). \tag{E.10}$$

Here we recall that $\ell \leqslant r$. Through induction on $\ell \geqslant 0$ and $\rho - \ell \geqslant 0$ using the recursion relation (E.8), we have

$$\Sigma(Z)^{\rho}_{\ell,m} = q^{-\ell m} \prod_{i=1}^{\ell} (1 - x q^{2m-2(\rho-i)}) \prod_{j=0}^{\rho-\ell-1} (1 - x q^{-2j}). \tag{E.11}$$

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